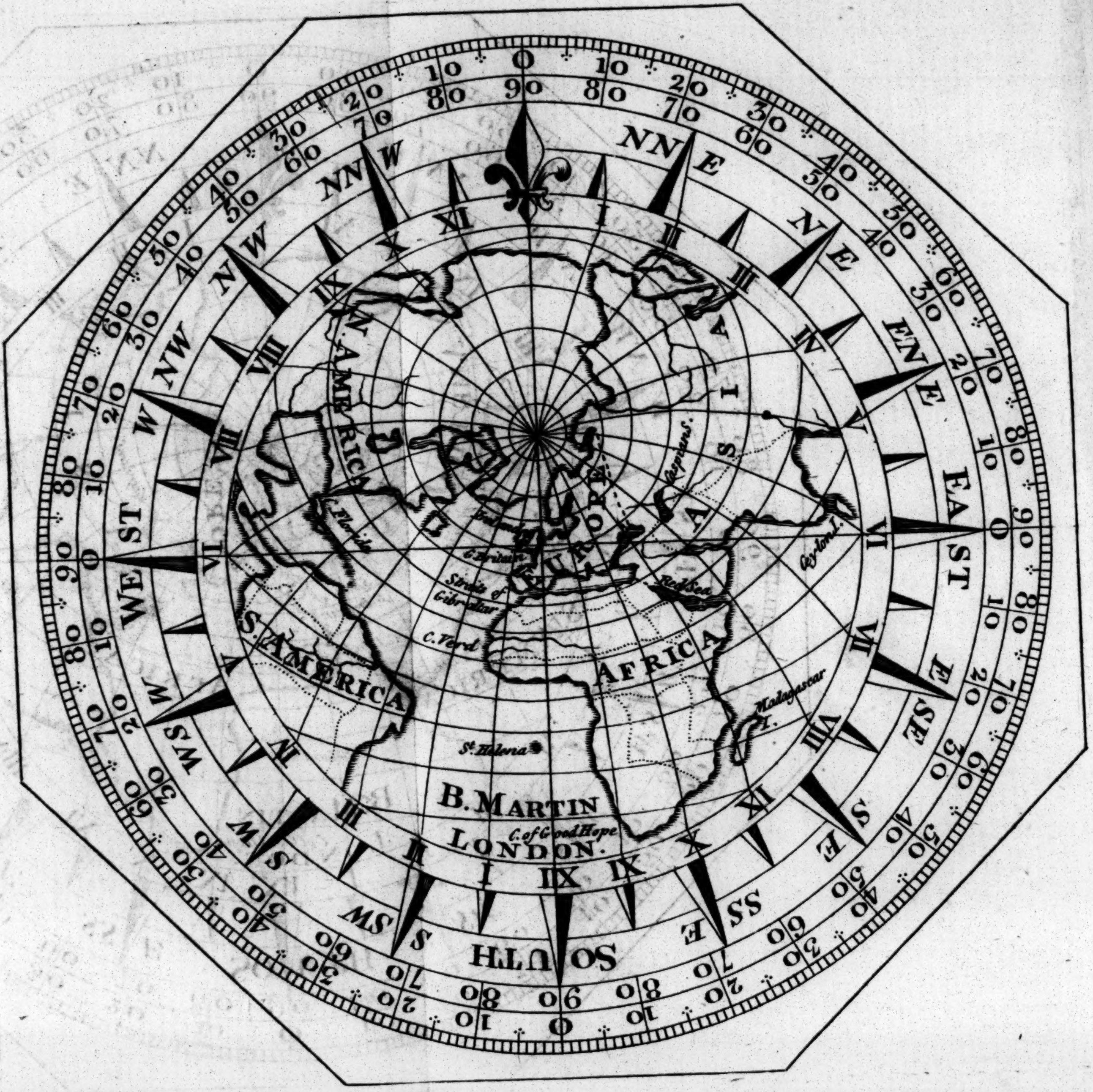B. MARTIN
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The Mariner's Mirror. — Part IV.

THE YOUNG TRIGONOMETER'S NEW GUIDE;

CONTAINING

The Elements of *Plain and Spherical* Trigonometry, deduced from the Principles of the Sphere in a Method entirely new; with the Analogies for ſolving all *Caſes*, and *Varieties* of *Triangles* in every Way hitherto invented; with their Application to Practice.

TOGETHER WITH

The Principles of *Fluxionary Trigonometry* fully explained and exemplified in many uſeful Problems.

ALSO

The Laws and Rules of the *Stereographic* Projection of the Sphere explained, and applied to the Delineation and Solution of *Spherical* Triangles, and the Conſtruction of Maps and Dials.

ILLUSTRATED BY FOUR COPPER-PLATES.

By B. MARTIN.

LONDON.
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B M

PREFACE.

THE ſame Reaſons induce me to epitomize and republiſh my YOUNG TRIGONOMETER'S NEW GUIDE, *as was aſſigned for doing the ſame by my* TREATISE *of* LOGARITHMS, *viz. the* conſtant Demand for it *ever ſince it has been out of Print, and the* many Improvements *that have been made in this Science, ſince the* firſt Publication *of this Work, which is now near* forty Years ago.

But as the Nature and Principles of all Sciences ever continue the ſame, the Improvements in each are chiefly to be expected from the MODES, *or different* METHODS *of* explaining *thoſe Principles, and of* applying *them to the Uſes of Life.*

In the firſt Compilation of this Work, I followed the common Routine *of Writers on this Subject in a tedious and prolix Demonſtration of numerous* THEOREMS *and* PROBLEMS *relative to* Plain and Spherical TRIGONOMETRY; *being little aware at that Time that there was a Path to this Field of Knowledge ſo ſtraight, ſo broad, ſo plain, and ſo ſhort, as I afterwards diſcovered in the* THEOREMS *of the learned* MAUPERTUIS; *and have here diſcloſed, elucidated, and applied to both Plain and Spherical Triangles in various Analogies deduced from theſe Theorems ſerving to the Solution of every poſſible Caſe.*

By theſe five Theorems here, I have demonſtrated more, and much more clearly, than I have done in forty before; and this affords a very great Saving *both in regard to the Expence of* Letter-Preſs *and* Copper-Plate *Print; for all the five Theorems are here derived from one Figure only, but many were neceſſary in my former Method; here the whole is tranſacted in twelve Pages, which employed a great Number there. Here nothing but a* COMPARISON *of two* ſimilar Triangles, *and the* SUM *and* DIFFERENCE *of two given Lines, are concerned in the demonſtrative Part; whereas in the former Work, all the* ELEMENTS *of* Plain *and* Spherical GEOMETRY *were barely ſufficient for the Purpoſe.*

After having formed and tabulated all Analogies, I proceed to illuſtrate the general Rationale *of Solution in all the*

the Variety of Caſes both of Plain *and* Spherical TRIANGLES *by proper Examples of Natural and Artificial* Sines, Tangents *and* Secants *as found in the* CANON *of* LOGARITHMS.

But though this might ſuffice for the intelligent Young Trigonometer, I have yet thought it expedient for Students in general, to give the APPLICATION *of* PLAIN TRIGONOMETRY *in the ſeveral Problems of* ALTIMETRY *and* LONGIMETRY; *and as I am well aſſured if theſe Examples are well underſtood, it will be quite unneceſſary to repeat the ſame over and over again in the ſame Triangles applied to other Arts, as in* Surveying, Navigation, Fortification, Gunnery, *&c. &c. which makes almoſt* half the firſt Volume *of my former Work.*

In like manner near half the ſecond Volume was taken up in the APPLICATION *of* Spherical TRIANGLES *to* Geography, Aſtronomy, Dialling, Great Circle Sailing, *&c. which amounted to little leſs than teaching thoſe ſeveral Sciences at large; but that is not my Deſign now; it muſt ſuffice to ſhew the Young Trigonometer how all the Triangles uſed in theſe Sciences are formed on the Surface of the Globe or Sphere, and then the Application of them to every Caſe that may occur, will be a proper Praxis or Exerciſe for him; and which he will find both delightful and eaſy.*

Not only the Solutions in Numbers, *but alſo by the* GUNTER, *or* Sliding-RULE, *as likewiſe by the* SECTOR, *are laid before him in the cleareſt and conciſeſt manner; and not only this, but he is inſtructed in the Uſe of a* New SCALE *of* NATURAL-SINES, *by which ſome Solutions are obtained to Caſes not reſolvable by either the* Sector *or* Gunter.

But the moſt valuable Improvement in this Science, we owe to the happy Genius of the late learned Mr. COTES, *which he publiſhed under the Title of the* ESTIMATION *of* ERRORS in mixed Mathematics; *of this Treatiſe I had never heard, when I compoſed my larger Work; nor have I ſeen it adopted ſince into any Treatiſe on this Subject, which is the more to be wondered at, becauſe it does in a moſt peculiar manner relate to the* Eſtimation and Correction of Errors, *ariſing from* trigonometrical Obſervations and Operations. *For this Reaſon I have given ample Specimens of the*

the Nature and Uſe of this Method in two Chapters under the Title of, The PRINCIPLES *of* Fluxionary TRIGONOMETRY, Plain and Spherical, *which I have illuſtrated by the celebrated Author's own Examples, and ſome others equally curious and intereſting. In ſhort, any competent Judge in theſe Matters, cannot but know, and will readily allow, that without theſe* new Fluxionary PRINCIPLES, TRIGONOMETRY *muſt be very* imperfect, *and its* Profeſſors *but* half-learned in their Art.

This EPITOME *of* TRIGONOMETRY *naturally follows the* COMPENDIUM *of* LOGARITHMS *which I lately publiſhed; and both together make the third and fourth Parts of the* Mariner's MIRROR; *for* TRIGONOMETRY *and* LOGARITHMS *are the two* PILLARS *upon which* NAVIGATION *wholly depends; and therefore it is, that I have not given particular numerical Examples for ſolving oblique ſpherical Triangles by theſe* Theorems *without a Perpendicular, as many ſuch may be found in the firſt and ſecond Parts of the ſaid* Mariner's Mirror, *to which I refer the Reader.*

To conclude, an Encomium *upon the* SUN *would not be more abſurd than one upon* TRIGONOMETRY, *as it is well known to be the entire Baſis of all* GEOMETRY, *and* every Mathematical SCIENCE. *The very firſt Propoſition of* EUCLID *is concerning a* TRIANGLE; *and the greateſt Diſcoveries in every Age have originated from a Contemplation and Compariſon of the Properties of thoſe uſeful Figures.*

Now if TRIGONOMETRY *be of ſuch Importance in itſelf, how highly muſt we conceive of its Uſefulneſs ſince the Invention and Application of* LOGARITHMS *thereto? But to give it its finiſhing Perfection, the all-compleating* SCIENCE *of* FLUXIONS *afforded it every Aſſiſtance it could poſſibly require; and in theſe moſt advantageous Circumſtances it is here preſented to the View of all whoſe Minds are delighted and captivated with uſeful and ſublime Improvements in Mathematical Sciences.*

And becauſe the Young TRIGONOMETER *may not complain of any Deficiency in this Treatiſe, I have added the Method of ſolving* Oblique Spherical Triangles, *by ſix* Subſidiary Right-angled Triangles, *from a foreign Anthor, on account of its* Novelty, Singularity, *and* Uſefulneſs.

Another curious Invention for the ſame Purpoſe I have likewiſe inſerted from a French Author, which by means of the Sums *and* Differences *of the* Sines, Co-ſines, *and* Tangents *of two Arches, ſolves all thoſe Caſes by the Trigonometrical Canon of Logarithms only.*

And leſt my young Pupil ſhould ſay, he is obliged in Verba Magiſtri jurare, *I have ſhewn him the Way to prove that the Value of every* Sine, Tangent, and Secant, *is the ſame as he finds them in the Natural Canon. This Satisfaction he may ſeek for in vain in other ſhort Tracts on this Subject.*

Theſe fifteen Chapters *contain all that I think properly belongs to the* Science *of* Trigonometry; *nor can the Tyro in theſe Studies be ſuppoſed ignorant of the Subject of any one of them with* Honour *to his* literary Character.

Here I ſhould have deſiſted, had it not been for a very cogent Reaſon, viz. That the Principles *of the* Stereographic Projection, *by reaſon of their extenſive Uſe in all the Branches of* Spherical Geometry, *and their* peculiar Affinity to the trigonometrical Part, *recommend themſelves ſo ſtrongly for the Completion of this Science, that I could not juſtify myſelf in denying my young Readers a* compendious View *and* Application *thereof, together with an Illuſtration by Examples in* Spherical Triangles, *and the* Construction *of* Maps *and* Dials. *And for this Purpoſe only, I have added the 4th Plate, as a Mathematical* Memento, *or* Remembrancer, *to the* Young Trigonometer, *in the Courſe of his Studies.*

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CHAP. I.

Elementary DEFINITIONS and ANALOGIES, for *Plain* and *Spherical* TRIGONOMETRY.

1. TRIGONOMETRY is the Art of computing the Quantity of the Sides and Angles of a TRIANGLE that are unknown, from others that are given or known. If the Triangle conſiſts of *Right-Lines*, it is called *Plain Trigonometry*; but if the Triangle be formed of three *Arches of great Circles*, it is then called *Spherical* TRIGONOMETRY.

2. In *Spherical* TRIGONOMETRY, the Quantity, or *Meaſure* of the Angles and Sides, conſiſts of a certain Number of *equal Parts* called *Degrees*, *Minutes*, *Seconds*, *Thirds*, &c. into which the Periphery of a Circle is ſuppoſed to be divided; this Diviſion of a Circle is into 360 Degrees; each Degree into 60 *Minutes*; each *Minute* into 60 *Seconds*, and ſo on in a *Sexagenary* Manner.

4. In like Manner, the Meaſure of the Angles of a plain or right-lined Triangle, is the *Arch of a Circle*;

but the Meaſure of its Sides are the *equal Parts* of *ſome ſtrait* LINE *or* SCALE, conſidered as a ſtandard Meaſure; as a *Pole*, a *Yard*, a *Foot*, an *Inch*, &c.

5. The RADIUS of a Circle, or Semicircle ABD (Fig. 1.) is half the Diameter AD, viz. AC, or CB, or CD; or it is the Length with which the Circle is deſcribed on the Center C.

6. According to the different Lengths of the Radius, the arch of the Circle which meaſures the ſame Angle will be proportionably different. Thus the Arch DF meaſures the Angle FCD to the Radius CD; and the Arch OP meaſures the ſame Angle to the Radius CP. And it is DF : OP :: CD : CP.

7. A CHORD is that Right Line which joins the two extreme Parts of the Arch of a Circle. Thus the Line IS is the *Chord* of the Arch IBS. It is alſo ſometimes called the *Subtenſe* of the ſaid Arch.

8. The SINE of an Arch is a right Line drawn from one End of the Arch perpendicular to the Diameter of the Circle paſſing through the other End; thus IV is the Sine of the Arch IB; IK is the Sine of the Arch IA; FG the Sine of the Arch FD. Note, the *Sine is half the Chord of twice the Arch*; thus IV is half IS, the Chord of IBS=2IB.

9. The CO-SINE of an Arch is the Sine of the *Complement* of that Arch to 90°, or a Quadrant; thus the *Complement* of the Arch DF is FB, the Sine of which is FH, therefore FH (=CG) is the Co-ſine of the Arch DF. Alſo FG is the Co-ſine of the Arch FB.

10. The *Verſed* SINE of an Arch is the Part of the Diameter of the Circle intercepted between the Sine and the Circle; thus GD is the verſed Sine of the Arch FD; and AG the verſed Sine of ABF, which is the

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the *Supplement* of the Arch FD to a Semicircle, or 180°.

11. The TANGENT of an Arch is a Right Line which touches the Circle in one Extremity of that Arch, and is cut by a Right Line drawn from the Center through the other Extremity of the Arch. Thus AM is the Tangent of the Arch AI; DE the Tangent of DF; and PR is the Tangent of the Arch OP or the Angle OCP to the Radius CP.

12. The SECANT of an Arch is a Right Line drawn from the Center C to interſect the Tangent of that Arch; thus CM is the Secant of AI; CE of the Arch FD; and CR of the Arch OP to the Radius CP.

13. The CO-TANGENT and CO-SECANT of an Arch, are the Tangent and Secant of the Complement of that Arch to 90°. Thus BN is the Co-tangent, and CN the Co-ſecant of the Arch DF.

14. The moſt conſiderable RELATIONS and PROPERTIES of theſe Lines are deducible from the ſimilar Triangles they compoſe in the Figure. Thus the ſimilar Triangles FCG and ECD give FG : CG : : ED : CD. i. e. *The Sine is to the Co ſine, as the Tangent is to the Radius.*

15. In the ſame Triangles we have FG : ED : : CF : CE. that is, *The Sine is to the Tangent, as Radius to the Secant.*

16. Alſo we have, CG : CD : : CF : CE. i. e. *Coſine is to Radius, as Radius to the Secant.*

17. Again, the ſimilar Triangles EDC, CBN, give ED : DC : : CB : BN. i. e. *The Tangent is to Radius, as Radius to the Co-tangent*

18. Hence becauſe AM : AC : : CB : BL, we ſhall have *The Tangent* ED *of one Arch* FD, *to the Tangent*

AM *of another Arch* AI; *as the Co-tangent* BL *of the latter to the Co-tangent* BN *of the former.*

19. The Triangles ECG, and CBN, give FG: CE :: CB : CN. i. e. *The Sine is to Radius as Radius to the Co-ſecant.*

20. Therefore alſo, becauſe IK: IC :: CB : CL; it will be, *As the Sine* FG : *the Sine* IK :: *Co-ſecant* CL *of the latter Arch* : *Co-ſecant* CN *of the former.*

CHAP. II.

The PRINCIPLES of *Spherical* GEOMETRY, in Fundamental THEOREMS.

21. LET P*p* be the AXIS of the SPHERE; AÆ*a* the EQUATOR; PA*pa* the MERIDIAN; HX*h* the HORIZON; Z the ZENITH; E the PLACE of a STAR; LE*l* a PARALLEL of ALTITUDE paſſing through it; DE*d* its PARALLEL of DECLINATION; ZE*z* the VERTICAL CIRCLE, and PE*p* the MERIDIAN, paſſing through the Star. (See Fig. 2.)

22. The Diameters of theſe Circles being drawn, and the Lines of the Interſections of their Planes, there will be formed the right-angled ſimilar Triangles BEF, CST; and CRV, GEF; on the inſide of the Sphere: and upon the Surface thereof, the Oblique Spherical Triangle PZE.

23. We ſhall denote the three Sides of this Triangle by the three Letters A, B, and C, as in the Figures 4 and 5; and the three Angles by the Letters P, Z, E,

which are placed by them. Alſo the *Radius*, *Sines*, and *Co-ſines*, of the Sides and Angles, are denoted by ſmall Letters, as in the Table below:

The Radius of the Sphere - - $r=CA, CH$, &c.

The Sine of the Arch AD - $x=BC=cs\,C$.
The Co-ſine - - - $y=BD=s\,C$.

The Sine of the Arch AZ - $s=PQ=cs\,B$.
The Co-ſine - - - - - $c=CQ=s\,B$.

The Sine of the Arch HL - $h=CG=cs\,A$.
The Co-ſine - - - $k=GL=s\,A$.

The Sine of the Angle ZPE - $t=ST=s\,P$.
The Co-ſine - - - $u=CT=cs\,P$.

The Sine of the Angle PZE - $m=RV=s\,Z$.
The Co-ſine - - - $n=CV=cs\,Z$.

24. It is evident from the Analogy (14) *That the Sine divided by the Co-ſine of any Arch, is equal to the Tangent of that Arch*, ſuppoſing Radius $r=1$. Alſo, *the Co ſine divided by the Sine is equal to the Co-tangent.*

Hence we ſhall have

$$\left\{\begin{array}{l} \frac{x}{y}=t\,AD, \text{ or } ct\,C. \\ \frac{s}{c}=t\,AZ, \text{ or } ct\,B. \\ \frac{h}{k}=t\,HL, \text{ or } ct\,A. \\ \frac{t}{u}=t\,ZPE, \text{ or } t\,P. \\ \frac{m}{n}=t\,PZE, \text{ or } t\,Z. \end{array}\right.$$

25. The Values of ſome elementary Lines are next to be determined in Terms of the *Sines* and *Co-ſines*, of

the Sides and Angles of the ſaid Triangle PZE, which is done by Analogies afforded by ſimilar Triangles in the following Manner. The ſimilar Triangles CST and BFE give the following Analogies, viz.

1. CS : ST :: BE : EF, that is, $r : t :: y : \frac{ty}{r} = \mathrm{EF}$.

2. CA : CT :: BD : BF, ——— $r : u :: y : \frac{uy}{r} = \mathrm{BF}$.

26. The ſimilar Triangles GFE and CRV give

1. CR : RV :: GE : EF, that is, $r : k :: m : \frac{mk}{r} = \mathrm{EF}$.

2. CH : CV :: GL : GF ——— $r : n :: k : \frac{nk}{r} = \mathrm{GF}$.

27. The ſimilar Triangles CPQ and CBO give

1. PQ : CP :: CB : CO, that is, $s : r :: x : \frac{rx}{s} = \mathrm{CO}$.

2. PQ : CQ :: CB : BO ——— $s : c :: x : \frac{cx}{s} = \mathrm{BO}$.

28. The Value of the Lines BO, CO, GF, EF, and BF, being known, we from thence raiſe the following *Five Fundamental* THEOREMS. It is evident that $\mathrm{GO} = \mathrm{GC} - \mathrm{CO} = \frac{hs - rx}{s}$; and the ſimilar Triangles FGO and PCQ give $c : r :: \frac{hs - rx}{s} : \mathrm{OF} = \frac{rsh - rrx}{cs}$. And (becauſe $\mathrm{BO} + \mathrm{OF} = \mathrm{BF}$) we have $\frac{ccx + rsh - rrx}{cs} = \frac{uy}{r}$ (25. 27.) That is (becauſe $cc - rr = -ss$) $\frac{rh - sx}{c} = \frac{uy}{r}$, and therefore $rrh - rsx = cuy$. Whence putting $r = 1$, we have

THEOREM I. $cs\mathrm{A} - cs\mathrm{B} \times cs\mathrm{C} = s\mathrm{B} \times s\mathrm{C} \times cs\mathrm{P}$.

29. The ſimilar Triangles PQC, FGO give $s:c::\frac{kn}{r}:GO=\frac{nck}{rs}$ (26.) Then (becauſe $CO+OG=CG$) we have $\frac{rrx+nck}{rs}=h.$ (27.) Therefore $rrx+nck=rsh.$ Hence

THEOREM II. $csC+csZ\times sB\times sA=csB\times csA.$

30. The ſimilar Triangles CRV, EFG give $m:n::\frac{yt}{r}:FG=\frac{nyt}{mr}$ (25.) And the Triangles QPC, FGO, give $s:r::\frac{nyt}{mr}:FO=\frac{nyt}{ms}.$ Then (becauſe $FO+OB=FB$) we have $\frac{nyt+mcx}{ms}=\frac{yu}{r}$ (25.) Hence $rynt+rmcx=msyu$; which gives in Sines and Tangents,

THEOREM III. $sP+tZ\times ctC\times sB=tZ\times csB\times csP.$

31. The ſimilar Triangles PCQ, FGO give $s:r::\frac{nk}{r}:FO=\frac{nk}{s}$ (26) Alſo $s:c::\frac{nk}{r}:GO=\frac{nkc}{rs}.$ And $CO=\frac{rsh-nkc}{rs}$ ($=CO-GO$;) Again the ſimilar Triangles PCQ, and CBO, give $r:c::\frac{rsh-nkc}{rs}:OB=\frac{rsch-nkcc}{rrs}.$ But $FO+OB:EF::\frac{nkrr+rsch-nkcc}{rrs}:\frac{mk}{r}$; which reduced, gives $rcht+nskt=rmku$; whence

THEOREM IV. $tP\times ctA\times sB+tP\times csZ\times csB=sZ.$

32. The common Section of the Almicantarath LE*l*, and the Parallel DE*d* deſcribed by the Star, give $\frac{mk}{r}=\frac{yt}{r}$ (25, 26.) Therefore $mk=yt.$ Whence

THEOREM V. $sZ\times sA=sC\times sP.$

CHAP. III.

The ANALOGIES for the RESOLUTION of all the CASES and VARIETIES of *Right Angled Spherical* TRIANGLES, derived from thoſe THEOREMS.

33. THE *Five* THEOREMS foregoing, comprehend almoſt the whole SCIENCE of *Spherical* GEOMETRY; at preſent we ſhall apply them to the Inveſtigation of ſuch Analogies as furniſh us with Solutions to the ſeveral Caſes of *Right Angled Spherical* TRIANGLES.

34. In order to this it muſt be conſidered, that any Angle Z of a Spherical Oblique Triangle PZE, may be ſo varied as to be *greater*, *equal to*, or *leſs* than a *Right Angle*; and in the Triangle before us (Fig. 2) it is evident, that when the Star E in its Parallel D*d* comes upon the *Prime Vertical* Zz, then the Angle at Z is a *Right one*. And the Triangle then becomes a *Right Angled one*, as ſhewn in Fig. 3. by PZE.

35. In this Caſe, it will be eaſy to ſee that the Sine of the ſaid Angle Z is equal to Radius, or $m=r=1$; and that the Co-ſine $n=o$, or vaniſhes. And conſequently in theſe Theorems the Terms multiplied by $n=cs\mathrm{Z}$, will vaniſh alſo; and $s\mathrm{Z}=1$, every where; and ſo the Theorems wherein the Angle Z is concerned, will ſtand as below:

Theorem II. $x=sh$ gives $cs\mathrm{C}=cs\mathrm{B}\times cs\mathrm{A}$.
III. $cx=syu$ $ct\mathrm{C}=ct\times \mathrm{B}cs\mathrm{P}$.
IV. $cht=ku$ $s\mathrm{B}\times ct\mathrm{A}=ct\mathrm{P}$.
V. $h=st$ $cs\mathrm{A}=s\mathrm{C}\times sP$.

36. Theſe Equations afford the following ANALOGIES, viz.

ANALOGY I. As Radius $1:sC::sP:sA$.
II. ——— $1::csB::csA:csC$.
III. —— $1:csP::ctB:ctC::tC:tB$ (18).
IV. —— $1:sB::tP::tA$.

37. To theſe we muſt add two more Analogies deducible from *Quadrantal Triangles* as follows. Let the Sides EZ, EP, ZP (Fig. 3.) be continued out till they become Quadrants in the Points N, F, and D; alſo continue the Arches PF, PD, till they become Quadrants in I and H. And then Arches of great Circles drawn through the ſaid Points N, F, D, and H, I, will interſect in G, and there will be formed the ſeveral *Quadrantal Triangles* NEF, ZDN, IPH, and IGF; in each of which there are two Right Angles oppoſed to two quadrantal Sides.

38. In theſe Triangles it is evident the Arch $NF=GD=E$; $HI=P$; the Angle $D=ZN$; therefore in the Triangle GHD, right angled at H, we have (by Analogy I.) $\text{Radius}:sGD::sD:sGH$, which in the Triangle EPZ, gives

ANALOGY V. $\text{Radius}=1:sE::csA:csP$.

39. Again in the Triangle PFD, right angled at F, we have (by Analogy IV.) $\text{Radius}:sPF::tHI:tDF$; this in the Triangle EZP, gives

ANALOGY VI. $\text{Radius}=1:csC::tP:ctE$.

40. In a Spherical Triangle there are ſix Parts, viz. *three Sides*, and *three Angles*; and as one of theſe Angles (viz. the *Right one* Z,) is always given, ſo if any two of the other five Parts be known or given, then by the above Analogies, the three unknown Parts will be found, as will appear by the following Table of Solu-

tions

tions to the three Caſes in each of the *Six Varieties* of the *Data* or Parts given.

41. The SOLUTION of Right Angled Spherical TRIANGLES.

Variety	Parts given	Sought	SOLUTION	Analogy
1	One Side A, and the adjacent Angle E.	B.	Rad. : *s*A : : *t*E : *t*B.	IV.
		C.	*cs*E : Rad. : : *t*A : *t*C.	III.
		P.	Rad. : *s*E : : *cs*A : *cs*P.	V.
2	A Side B. and oppoſite Angle E.	A.	*t*E : *t*B : : Rad. : *s*A.	IV.
		C.	*s*E : *s*B : : Rad. : *s*C.	I.
		P.	*cs*B : Rad. : : *cs*E : *s*P.	V.
3	Hyp. C. One Angle E.	A.	Rad. : *cs*E : : *t*C : *t*A.	III.
		B.	Rad. : *s*C : : *s*E : *s*B.	I.
		P.	*cs*C : Rad. : : *ct*E : *t*P.	VI.
4	Two Sides A and B.	C.	Rad. : *cs*A : : *cs*B : *cs*C.	II.
		E.	*s*A : Rad. : : *t*B : *t*E.	IV.
		P.	*s*B : Rad. : : *t*A : *t*P.	IV.
5	Hyp. C. Side B.	A.	*cs*B : Rad. : : *cs*C : *cs*A.	II.
		E.	*s*C : Rad. : : *s*B *s*E.	I.
		P.	*t*C : Rad. : : *t*B : *cs*P.	III.
6	Two Angles E and P.	A.	*s*E : *cs*P : : Rad. : *cs*A.	V.
		B.	*s*P : *cs*E : : Rad. *cs*B.	V.
		C.	*t*P : *ct*E : : Rad. *cs*C.	VI.

CHAP.

CHAP. IV.

An INVESTIGATION of the ANALOGIES for ſolving all the VARIETY of CASES of *Oblique Spherical* TRIANGLES.

42. AN oblique Spherical Triangle EZP (See Fig. 4 and 5.) having three Parts known, the other three may be found by Means of the foregoing *ſix Analogies*, and *four more* to be inveſtigated from them in the following Manner.

43. From an Angle P let fall the Perpendicular PQ upon the oppoſite Side EZ (continued out, if required) then there will be formed two right-angled Spherical Triangles PQE and PQZ, which (by Analogy III.) give the two following Proportions:

$$\text{viz.} \left\{ \begin{array}{l} \text{Radius} : cs\,QPE :: t\,EP : t\,QP. \\ \text{Radius} : cs\,QPZ :: t\,ZP : t\,QP. \end{array} \right\} \text{ therefore we}$$

have

$$\text{ANALOGY VII. } tC : tB :: cs\,QPZ : cs\,QPE.$$

44. In the ſame Triangles we have (by Analogy II.)

$$\text{The Ratios} \left\{ \begin{array}{l} \text{Radius} : cs\,QP :: cs\,QE : cs\,EP. \\ \text{Radius} : cs\,QP :: cs\,QZ : cs\,PZ. \end{array} \right.$$

$$\text{Hence ANALOGY VIII. } cs\,QE : cs\,EP :: cs\,QZ : cs\,PZ.$$

45. Again (by Analogy V.) we have theſe Proportions:

$$\text{viz.} \left\{ \begin{array}{l} \text{Radius} : s\,EPQ :: cs\,QP : cs\,E. \\ \text{Radius} : s\,ZPQ :: cs\,QP : cs\,Z. \end{array} \right.$$

$$\text{Hence ANALOGY IX. } cs\,E : cs\,Z :: s\,EPQ : s\,ZPQ.$$

46. Laſtly; we have (by Analogy IV.) theſe

$$\text{Proportions}\begin{cases}\text{Radius} : sQE :: tE : tQP.\\ \text{Radius} : sQZ :: tZ\ tPQ.\end{cases}$$

Hence ANALOGY X. $sQE : sQZ :: tZ : tE$.

47. The laſt general Theorem (32.) furniſhes us with the *Proportionality of the Sines of the Sides to the Sines of their oppoſite Angles*, viz.

ANALOGY XI. $sA : sP :: sC : sZ$.

48. In Caſe the Angle Z be *obtuſe*, or greater than 90°, (as Fig. 5.) then $EQ-ZQ=EZ$, and $EPQ-QPZ=EPZ$; but the Angle Z being *acute* (as in Fig. 4.) the Perpendicular PQ falls within the Triangle, and then the Side $EZ=EQ+QZ$; and the Angle $EPZ=EPQ+QPZ$.

49. In the Solution of the Triangle, the Perpendicular PQ muſt be let fall from the End of a given Side, and oppoſite to a given Angle.

50. There are in an oblique Spherical Triangle alſo *Six Varieties* in the *Data*, the four firſt of which have all their Caſes ſolved by the Analogies above; and the fifth is ſolved by Theorem I. (28.)

51. The ſixth Variety is alſo reſolvable by the ſame Theorem, becauſe the *Angles of the Triangle* EPZ (Fig. 2.) *are equal to the* Sides of another Triangle CIK, ſuppoſing C, I, K, the Poles of the great Circles PA*p*, ZE*z* and PE*p*; becauſe the Inclination of the Planes of any two Circles is equal to the Diſtance of their Poles; therefore the Angle $P=CK$, the Angle $Z=CI$, and the Angle $E=IK$. So that the ſame Theorem equally ſerves for the Solution of the fifth and ſixth Variety.

52. The SOLUTION of Oblique Spherical TRIANGLES.

Variety	Given	Sought	SOLUTION	Analogies
1	Two Angles E, Z, and an oppoſite Side B.	C	$sE : sB :: sZ : sC.$	XI.
		A	$\left\{\begin{array}{l} \text{Rad.} : csZ :: tB : tQZ \\ tE : tZ :: sQZ : sQE \end{array}\right\} QE \mp QZ = A.$	III. X.
		P	$\left\{\begin{array}{l} csB : \text{Rad.} :: ctZ : tQPZ \\ csZ : csE :: sZPQ : sEPQ \end{array}\right\} \begin{array}{l} QPZ \mp EPQ \\ = P. \end{array}$	VI. IX.
2	Two Angles E, P, and Side included C.	Z	$\left\{\begin{array}{l} csC : \text{Rad.} : ctE : tEPQ. \\ \text{Then } EPQ \pm EPZ = QPZ\ ; \\ \text{Then } sEPQ : sZPQ :: csE : csZ. \end{array}\right.$	VI. IX.
		B	$\left\{\begin{array}{l} \text{Find the Angle } QPZ, \text{ as above.} \\ \text{Then } csQPZ : csQPE :: tC : tB. \end{array}\right.$	VII.
		A	This is found in the ſame Manner by a Perpendicular from E to PZ, continued out.	
3	Two Sides C and B, and an oppoſite Angle E.	A	$\left\{\begin{array}{l} \text{Rad} : csE :: tC : tQE \\ csC : csB :: csQE : csQZ \end{array}\right\} QE \mp QZ = A.$	III. VIII.
		P	$\left\{\begin{array}{l} csC : \text{Rad.} :: ctE : tEPQ \\ tB : tC :: csEPQ : csZPQ. \end{array}\right\} \begin{array}{l} EPQ \mp ZPQ \\ = P. \end{array}$	VI. VII.
		Z	$sB : sE :: sC : sZ.$	XI.
4	Two Sides A and C, and the included Angle E.	B	$\left\{\begin{array}{l} \text{Rad.} : csE :: tC : tEQ \therefore EQ \mp EZ = QZ \cdot \\ cs\ QE : csQZ :: csC : csB. \end{array}\right.$	III. VIII.
		Z	$\left\{\begin{array}{l} \text{Find } ZQ, \text{ as above. Then } sZQ : sQE :: \\ \quad tE : tZ. \end{array}\right.$	X.
		P	This is found in the ſame Manner by letting fall a Perpendicular from Z, to the Side EP.	
5	All the 3 Sides A, B, C.	P	$\dfrac{csA - csB \times csC}{sB \times sC} = csP.$ by $\left\{\begin{array}{l} \text{Theorem I.} \\ \text{at Art. 28.} \end{array}\right.$	
		E	$sA : sP :: sB : sE.$	XI.
		Z	$sA : sP :: sC : sZ.$	XI.
6	The 3 Angles, P, Z, E.	A	Conſider all the Angles as Sides. Then $\dfrac{csP - csE \times csZ}{sE \times sZ} = csA$, Theorem I.	

53. I might here ſhew how the Caſes of the four firſt Varieties might be ſolved by the *five fundamental* THEOREMS, without a Perpendicular; but ſhall wave that Matter for the preſent.

54. It muſt be remembered, that only one of the five fundamental Theorems has a *negative* Sign; viz. the firſt (at 28.) and that ſince $GC-CO=\frac{hs-rx}{s}$, the Signs will be different when the Almicantharah L*l* cuts CZ below the Point O, for then it will be $GO=CO-GC=\frac{rx-sh}{s}$.

55. Every right-angled Spherical Triangle has its *Supplemental one*; and therefore when *an Angle*, and *a Side oppoſite to it* are given, the *Baſe* and *Hypothenuſe* ſought will be ambiguous, becauſe they may be thoſe of one Triangle as well as of the other, the *Sine of an Arch and of its Supplement being the ſame.* (Art. 10.)

56. Alſo in an Oblique Triangle PZE, all the three Caſes of Variety III. will be ambiguous when the Side PZ is leſs than the Side EP; for the Angle Z may be acute or obtuſe; and the other Angle P, and Side A will be of different Quantities, as they relate to the Fig. 4 or 5, as is evident from a bare View of them.

CHAP.

CHAP. V.

The ANALOGIES for the SOLUTION of all the CASES of *Right-angled* PLAIN TRIANGLES.

57. THE Solution of a *plain right-angled* TRIANGLE depends entirely upon that of a *Spherical one*, as is evident from hence, that ſuch a Spherical Triangle, by ſuppoſing the *Radius of the Sphere infinite*, becomes a *plain* or *right-lined* TRIANGLE; ſince in that Caſe, the verſed Sine GD of any Arch DF vaniſhes; and the Sine GF and Tangent DE do both coincide with the Arch itſelf, which then becomes a *Right Line*. (See Fig. 1.)

58. It is alſo evident, that ſince *the three Angles of a Plain Triangle are equal to two Right Angles*, if one of them be a Right Angle, as in the preſent Caſe, the other two will be together equal to 90 Degrees; therefore if one Angle in a right-angled Triangle be given, they are all given; but yet they are not ſufficient *Data* for the Solution of the Triangle, ſince all ſimilar Triangles have the ſame equal Angles.

59. Another thing to be obſerved, is, that *the Square of the Hypothenuſe is equal to the Sum of the Squares of the Legs or Sides*; thus $C^2 = A^2 + B^2$. (Fig. 6.) and therefore when two of theſe are given, they find the third.

60. Upon theſe Conſiderations it will appear, that with regard to a right-angled Plain Triangle, there can be but *Four Varieties* in the *Data*, and but two Caſes in each; and that the *Canons* for their Solutions are the ſame as thoſe for the ſame Caſes of a Spherical Triangle, contained in the Table at Art. 41. if for the Sine or Tangent of a Side we take the Side itſelf. All which will be evident, by comparing the Solutions there with thoſe in the following Table, for a *plain right-angled* TRIANGLE.

61. The SOLUTION of Right-angled Plain TRIANGLES.

Variety	Given	Sought	SOLUTION
1	The Hypoth. C, and the Angles P and E.	A B	$\text{Rad.}\ C :: sP : A.$ $\text{Rad.}\ C :: sE : B.$
2	Hypoth. C, and 1 Side B.	E A	$C : \text{Rad.} :: B : sE.$ $\text{Rad.} : C :: sP : A.$ Or thus $\sqrt{C^2-B^2}=A.$
3	The Angles P and E, and Side B.	C A	$sE : B :: \text{Radius} : C.$ $sE : B :: sP : A.$ Or $\text{Rad.} : B :: tP : A.$
4	Two Sides A and B.	E C	$A : \text{Rad.} :: B : tE.$ $sE : \text{Rad.} :: B : C.$ Or $\sqrt{A^2+B^2}=C.$

62. In the Solution of the 2d Caſe of the 3d Variety, you have two Analogies, one in *Sines* of the Angles, and the other by *Radius* and *Tangent* of an Angle;

which latter is preferable in Practice, as it requires but *one Operation*, but the other, *two*; as we ſhall ſhew farther on. The Reaſon of this, is evident from Fig. 6. wherein, by making the Hypothenuſe EP Radius, and deſcribing the Arch P*b*, it is plain the Side B is the Sine of the Angle E, and A the Sine of the Angle P, and therefore *s*E : B : : *s*P : A. But by making EZ (or the Side A) Radius, and deſcribing the Arch Z*a*, it is evident, the Side B is the Tangent of the Angle E; and therefore, Radius : A : : *t*E : B; and for the ſame Reaſon by making ZP (or B) Radius, you have Radius : B : : *t*P : A.

63. Alſo by making EZ (or A) Radius, the Hypothenuſe EP (or C) will be the *Secant* of the Angle E, and then we have Radius : A : : *se.* E : C. Or Radius : B : : *se.* P : C. All which is evident from the Definitions of *Sines*, *Tangents*, and *Secants*, in Chap. I. and the *Proportionality of Plane ſimilar Triangles.*

CHAP. VI.

The SOLUTION of *Oblique Plain* TRIANGLES.

64. THE Sides of an Oblique Spherical Triangle become *right Lines* upon Suppoſition of an *infinite Radius*; and conſequently the Triangle itſelf becomes a *plain oblique Triangle*, wherein all the Angles are *acute*, as Fig. 7. or having one Angle Z *obtuſe*, as Fig. 8.

65. Hence *Analogy* XI. (Art. 47.) $sA : sP :: sC : sZ$: becomes $A : sP :: C : sZ$, or the *Sides of the Triangle are proportional to the Sines of the oppoſite Angles.*

66. When two Angles are given, the third is known, as being the Supplement to 180°. But though the three Angles are known, they do not determine the Magnitude, or Length of any Side of the Triangle, as it does not depend upon them.

67. The *Sum* of two Quantities ($a+b=s$) and their Difference ($a-b=d$) being given, the Quantities are from thence given alſo; for thoſe two Equations give $b=s-a=a-d$; and therefore $s+d=2a$. Whence the greater Quantity $a=\frac{s+d}{2}$; and in like Manner the leſſer $b=\frac{s-d}{2}$.

68. In any oblique Triangle ABC (Fig. 9.) produce one Side BC to E, and make $CE=CA$; and biſect BE in D. Join AE, and drawing DF parallel to AB, it will biſect AE in F. Alſo draw CG parallel to AB. Then is the external Angle $ACE=B+BAC=2ACF$, and the Angle $GCA=BAC$; therefore if from *half the Sum* ACF ($=\frac{1}{2}A+\frac{1}{2}B$) you take the leſſer Angle $A=ACG$, there will remain *half their Difference* $\frac{1}{2}B-\frac{1}{2}A=GCF$. Alſo from *half the Sum* of the Sides $\frac{1}{2}AC+\frac{1}{2}BC=BD$, you take the leſſer Side BC, there will remain half their Difference $\frac{1}{2}AC-\frac{1}{2}BC=CD$. On the Center C, with Radius CF, ſtrike the Arch *aFb*, then is AF the Tangent of the Angle ACF, and GF the Tangent of GCF; and by ſimilar Triangles we have $BD : CD :: AF : GF$, which gives the following

ANALOGY XII. *As the Sum of the Sides is to their Difference, ſo is the Tangent of half the Sum of the oppoſite Angles, to the Tangent of half their Difference.*

69. When an oblique Triangle ABC (Fig. 10.) has its three Sides given, it muſt be reduced to *two right angled Triangles* ADB, and ADC; and the beſt Method of doing this, is the following one of Sir ISAAC NEWTON. Let $AB=a$, $AC=b$, $BC=c$, and $BD=x$; then $DC=c-x$. Now ſince $AB^2-BD^2=(a^2-x^2)=AD^2$; and $AC^2-DC^2=(=b^2-c^2+2cx-x^2)$ AD^2; it will be $a^2-x^2=b^2-c^2+2cx-x^2$, conſequently $\frac{a^2-b^2+c^2}{2c}=x$, or $\frac{AB^2-AC^2+BC^2}{2BC}=BD$; then DC is known; and thus two Sides are known in each Triangle, to find the Angles, by the Caſes of Variety IV. (41.)

70. With regard to oblique Triangles, there are alſo four Varieties in the *Data*, the Caſes of which are ſeverally ſolved by the Articles above premiſed. The Solutions of thoſe of the ſecond Variety will be ambiguous, when the Side AB oppoſite the given Angle C is leſs than the given Side AC. (See Fig 11.) adjacent to it; for let B*a* be drawn equal to BA; then will the *Data* belong equally to the obtuſe-angled Triangle B*a*C, and the acute Triangle BAC, and whether it be the Angle A or *a*, or the Side AC or *a*C, that is found, muſt be determined from the Nature and Conditions of the Queſtion.

71. The SOLUTION of *Oblique Plain* TRIANGLES.

Variety	Given	Sought	SOLUTION
1	The Angles, and a Side A B.	BC AC	$sC : AB :: sA : BC$. (65.) $sC : AB :: sB : AC$.
2	Two Sides AB, BC; and an oppoſite Angle C.	A B AC	$AB : sC :: BC : sA$. Then $C+A-180=B$. $sC : AB :: sB : AC$.
3	Two Sides AC, AB; and the included Angle A.	B C BC	$AB+AC : AB-AC :: t\tfrac{1}{2}\overline{B+C} : t\tfrac{1}{2}\overline{B-C}$ By the half Sum, and half Difference, the Angles B and C are known. (See Art. 67, 68.) $sC : AB :: sA : BC$.
4	All the three Sides.	BD	$\frac{AB^2-AC^2+BC^2}{2BC}=BD$. (69.)

N. B. In theſe oblique Plain Triangles, the Sides are denoted, as uſual, by the two Letters at the Ends, for the Sake of greater Congruity of this Synopſis with the preceeding Articles.

C H A P. VII.

The *Rationale* of *Trigonometrical* PROCESSES, and CALCULATIONS; with, and without LOGARITHMS.

72. WHEN two Sides of a plain right-angled Triangle are known, the third Side may be found without the *Trigonometrical* CANON (59.) but no Solution for the Angles, nor for the Caſes of any

other Variety, can be had without having Recourſe to the ſaid Canon, for this Canon conſiſts wholly in the numerical Expreſſion of the Value of the *Sine*, *Tangent*, and *Secant*, of every Minute of the Quadrant in ſuch Numbers or Parts of which the Radius contains 1,000000.

73. Let the ſaid Tabular Triangle be that large one QCO (Fig. 1.) and let the Angle at $C=33^\circ\ 15'$.

Then the *Radius* being - $CP=CO=1,000000$
the *Sine* will be - - - $OQ=548293$
the *Tangent* - - $PR=655629$
the *Secant* - - - $CR=1,195763$
Alſo the *Sine* of the Complement $CQ=56^\circ\ 45'$ 836286
the *Co-tangent* - - - 1,525253
the *Co-ſecant* - - - - 1,823842

74. By this Artifice there is always a Triangle ready calculated in the Tables, and ſimilar to any one that can be propoſed for Solution; and ſince in every Analogy there are two Terms (of the four) in the tabular Triangle, and two in the given one, of which one is known, it is plain there are three Terms known to find the fourth, by the common Rule of Three.

75. For Example; let the propoſed Triangle be FGC, in which there are given the Baſe $CG=165$, and the Angle at $C=33^\circ\ 15'$, to find the Perpendicular GF. Then ſimilar to this is the tabular Triangle CQO, which having the ſame Angles $C=33^\circ\ 15'$ and $O=56^\circ\ 45'$, and making CO Radius, we have by the Canon of Natural Sines, $CQ=836286$, and $OQ=548293$; whence the following Analogy:

$$\text{As}\left\{\begin{array}{ccccccc} CQ & : & QO & :: & CG & : & GF. \\ 836286 & : & 548293 & :: & 165 & : & 108,2. \end{array}\right.$$

76. If inſtead of making CO Radius, and working by Sines, you make CQ Radius, then the two Sides

CQ and OQ which before were *Sines*, now become *Radius* and *Tangent*; and the Operation for finding the Side GF will become more ſimple and eaſy, for then the Analogy will be,

As { CQ : QO : CG : GF.
{ 1000000 : 655629 : 165 : 108,2.

77. But as the Operation by Natural Sines and Tangents, in common *Numbers*, is very troubleſome and laborious; theſe Trigonometrical Proportions are now wrought, where Exactneſs is required, by *artificial Numbers* or *Logarithms*, whoſe Property it is to perform *Multiplication* and *Diviſion* by the *Addition* and *Subtraction* of two Lines of Figures only.

78. Thus by taking the Logarithmic Sines, the Example of Art. 75 will ſtand as follows:

	Logarithms
As the Sine of the Angle COQ=56° 45′	9,922355
Is to the Sine of the Angle C=33° 15′ -	9,739013
So is the Side CG=165 -	2,217484
Sum	1,956497
To the Side required GF=108,2 -	2,034142

Here the whole Operation conſiſts in adding the Logarithms of the 2d and 3d Numbers, or Terms of the Analogy, and from that Sum ſubducting the Logarithm of the firſt Term, there will then remain the Logarithm of the Number which expreſſes the Quantity ſought, and which is found in the Canon.

79. If we work with Radius and Tangent, as in the Example of Art. 76, we ſhall find the Operation ſtill much eaſier, even in Logarithms; becauſe now the firſt Term being Radius, its Logarithm 10,000000 gives no Trouble in ſubtracting; the whole, in Effect, being

performed by Addition of the Logarithms of the 2d and 3d Terms only, as is ſeen in the Work below:

As the Radius - - - 10
Is to the Tangent of the Angle C – 33° 15′ 9,816658
So is the Baſe CG = 165 - 2,217484

To the Perpendicular ſought GF = 108,2 2,034142

80. Again, ſuppoſe the Side or Hypothenuſe CF were ſought; then if CO be made Radius, it will come into the Operation with the Sine of the Angle at O, as follows:

As the Sine (CQ) of the Angle O = 56° 45′ 9,922355

Is to Radius (CO) - - 10,
So is the Baſe CG = 165 - - 2,217484

To the Hypothenuſe CF = 197,3 - - 2,295129
Here the whole Operation is in *one Subtraction only.*

81. But if CQ be Radius, it will be uſed with the *Secant* of the Angle at C, to find the Hypothenuſe CF, as in the following Operation:

As the Radius (CQ) - 10,
To the Baſe CG = 165 - 2,217484
So is the Secant (CO) of 33° 15′ - 10,077645

To the Hypothenuſe CF = 197,3 - 2,295129

82. Theſe are all the Varieties in the Solution of Caſes of Triangles, whether they are plain or ſpherical, right or oblique angled; and they who underſtand what has been already ſaid, will need no further Inſtruction; they will alſo obſerve, that the laſt Proceſs with *Secants* is ſuperfluous; as no Analogies in the foregoing Tables (41, 52, 61, 71) include or mention a *Secant*, and therefore alſo moſt Canons or Tables of Logarithms leave them out. But as Secants are on ſome Occaſions wanted, they may be eaſily had from

the Canon of Sines; *for the Logarithm of the Co-ſine ſubducted from twice the Logarithm of the Radius, will leave the Secant required*, as is evident from Art. 16.

83. The firſt Proceſs in Logarithms (at 78) may be performed by *Addition only*, by taking the *Arithmetical Complement* of the Logarithm of the firſt Term of the Analogy, which is had by taking every Figure of the Logarithm from 9, but the laſt, which is taken from 10, beginning at the left hand. Thus the *Arithmetical Complement* of the Logarithm 9,922355 is 0,077645; and this Operation being performed *mentally*, is by ſome preferred in Practice, becauſe this *Arithmetical Complement* of the firſt being added to the Logarithms of the ſecond and third Terms, give the Logarithm of the fourth; as in the Example below:

Add	Arith. Comp. of Sine 56° 45′ -	0,077645
	Logarithm of the Sine of 33° 15′	9,739013
	Logarithm of the Side CG=167	2,217484
	The Sum is the Logarithm of CF=108,2	2,034142

The ſame as before (78.) with 4 Lines of Figures inſtead of 5, and by *Addition only*.

C H A P. VIII.

The APPLICATION of *Plain* TRIGONOMETRY to ALTIMETRY and LONGIMETRY.

84. ALTIMETRY is the Art of *meaſuring the* HEIGHTS of Objects *trigonometrically*; and LONGIMETRY is that of *meaſuring their* DISTANCES;

and this in all Caſes, whether they are, or are not, *acceſſible*. The common Inſtruments for this Purpoſe are good QUADRANTS, PLAIN TABLES, or THEODOLITES, by which the Angles, thoſe Heights and Diſtances ſubtend at the Eye, are meaſured. This Doctrine will be illuſtrated by the following Examples.

EXAMPLE I.

85. Let it be required to meaſure the Height of an acceſſible Object, as the Tree BC. (Fig. 12.)

Meaſure any convenient Diſtance from the Body of the Tree B to A, which ſuppoſe to be 127 Yards, then at A, with a Quadrant, take the Angle of Altitude BAC, which ſuppoſe to be $36^{\circ}\ 30'$, whoſe Complement is $53^{\circ}\ 30 = BCA$. Then ſay, (by Variety III. at 61.)

	* A. C.
As the Sine of the Angle $C = 53^{\circ}\ 30'$ -	0,094821
Is to the Diſtance $AB = 127$ - -	2,103804
So is the Sine of the Angle $A = 36^{\circ}\ 30'$ -	9,774388
To the Height of the Tree above the Eye, $BC = 93{,}97$	1,973013

EXAMPLE II.

86. To meaſure the Height of an Object, as the Spire BC, by its Shadow and a Staff (Fig. 13.)

Meaſure the Shadow of the Spire BA, which ſuppoſe to be 150 Feet; then at the End of the Shadow A ſet the ſtaff (bc) perfectly upright, which ſuppoſe is three Feet, then meaſure its ſhadow (ba) which let be four Feet. Then becauſe of *ſimilar Triangles* ABC and abc, we have $ab = 4 : bc = 3 :: AB = 150 : BC = 112{,}5$; the Height of the Spire required.

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* Note A. C. ſtands for *Arithmetical Complement.*

EXAMPLE III.

87. To meaſure the Height of an *inacceſſible Object*, as the Tower BD, which you cannot approach by reaſon of the River R. (Fig. 14.)

At any convenient ſtation C, take the Angle BCD=55° 30′; then from C meaſure any Diſtance as CA=108 Yards; and at A take the Angle DAB=40° 00′. Then in the oblique Triangle CBA, the Angles are known, and one Side AC, to find the Side BC (by Variety I. at 71.)

	A. C.
Thus, As the Sine of the Angle ABC= 15° 30′	0,573101
Is to the Length of AC=108	2,033424
So is the Sine of the Angle BAC=40° 00′	9,808067
To the Length of the Side CB=259,7	2,414592

88. Hence in the right-angled Triangle CBD there are given the Angles, and the Hypothenuſe BC to find the Sides BD and DC (by Variety I. at 61.)

Thus, As Radius	10,
Is to the Side BC=259,7	2,414592
So is the Sine of the Angle DCB=55° 30′	9,915994
To the Height of the Tower DB=214	2,330586
And, ſo is the Sine of the Angle CBD=34° 30′	9,753128
To the Diſtance thereof CD=147,1	2,167720

EXAMPLE IV.

89. To meaſure the Diſtance of an inacceſſible Object, as the Windmill B, beyond the River R. (Fig. 15.)

By the Plain Table or Theodolite, the Angles at A and C are meaſured, and by the Chain meaſure the

Side AC; then in the oblique Triangle ABC you have given two Angles and the Side included, to find the Side CB, (as in Example III.) which is the Diſtance required.

EXAMPLE V.

90. To find the Diſtance between two Objects B, *b*, inacceſſible by the River R. (Fig. 16.)

Let the Diſtance AC be meaſured, and the Angles CAB and ACB be taken by the Theodolite; and the Side AB, or Diſtance of the Houſe B, is known, by Example III. Alſo take the Angles AC*b* and CA*b*, and then in the Triangle CA*b*, the Side A*b* is found, or Diſtance of the Tree at (*b*). Then, laſtly, having the two Sides AB, A*b*, and the included Angle BA*b* by the Theodolite, you find the third Side B*b* (by Variety III. at 71)

EXAMPLE VI.

91. To find the Height of the Tower and Spire B*b* ſituated upon the Hill EBD. (Fig. 17.)

Meaſure a proper Diſtance from C to A with the Pole or Chain; and at A and C take with a Quadrant the Angles *b*AD and *b*CD; alſo the Angles BAD and BCD; then in the oblique Triangle A*b*C you find the Side C*b*; and in the Triangle ACB you find the Side CB; and taking the Angle BC*b*, you find the Side B*b*, (as in the laſt Example) the Height of the Tower required.

92. It is plain from theſe Examples, that *Altimetry* and *Longimetry* are the ſame Thing in Effect and Operation; and, in ſhort, all the Operations in *Surveying*, *Navigation*, *Gunnery*, *Fortification*, *Mechanics*, &c. conſiſt in the Solutions of a Plain Triangle; and therefore

to exemplify in each particular, would only be a needleſs Repetition of the above Examples, over and over again.

CHAP. IX.

The Practical METHOD of ſolving all CASES of TRIANGLES by the *Single* or *Sliding* GUNTER; by the SECTOR; and by the *New Univerſal* SLIDING RULE.

93. IN practical Caſes, where great Accuracy is not required, we generally make Uſe of the GUNTER, as it is called, that is, a Set of *artificial* LINES of *Numbers*, *Sines*, and *Tangents*, which are nothing but the Logarithms of the Canon laid off upon *ſtrait Lines*, firſt invented by Mr. *Edmund Gunter*, whoſe Name they bear. Theſe Lines are uſed either *ſingly* with *Compaſſes*, or *by Pairs*, on a *ſliding* RULE; where one is fixed upon the Rule, and the other moveable by it upon the Slide.

94. The Uſe of the *Single* GUNTER with Compaſſes, is this: Suppoſe the ſame Parts given in the plain right-angled Triangle CGF (Fig. 1.) as before (75) viz. the Side CG=165; the Angle C=33° 15′, and the Angle at O=56° 45′, to find the Side FG the Operation is thus: Set one Foot of the Compaſſes in 56° 45′ of the *Line of Sines*, and extend the other to 33° 15′; then with that extent place one Point of the Compaſſes in 165 in the *Line of Numbers*, and turning the other about it will fall upon 108,2=FG, the Side ſought (as at 78.)

95. Or by uſing the *Line of Tangents* (according to Art. 79.) you proceed thus: Set one Point of the Compaſſes in 45° (the Radius) of the *Line of Tangents*, and extend the other to 33° 15′, with this Opening of the Compaſſes place one Foot in 165 in the *Line of Numbers*, the other will reach to 108,2 in the ſame Line, which is the Side GF required.

96. With the *ſliding Gunter* you work the above Solution in the following Manner; and firſt with the *Lines of Numbers and Sines*, thus: Set 56° 15′ on the Slider to 33° 15′ on the Rule, (in the *Lines of Sines*) then againſt 165 on the Slide is 108,2 on the Rule, in the *Lines of Numbers*. Or ſet 56° 15′ on S to 165 on N, then againſt 33° 15′ on S is 108,2 on N for the Anſwer. Note, the *Line of Numbers*, *Sines*, and *Tangents*, are marked with N, S, and T at the Ends.

97. The ſame Proportion is wrought with the *Tangents*, thus; Set Radius (viz. 45°) on the Slide to 33° 15′ on the Rule; then againſt 165 on the Slide is 108,2 on the Rule, on the Lines N. Or thus: Set Radius on T to 165 on N, then againſt 33° 15′ on T is 108,2 on N.

98. To find the Hypothenuſe CF, the Analogy (at Art. 80.) is thus performed: Set 56° 45′ to Radius (90°) on the Sines S, S; then againſt 165 is 197,3 on the Lines N, N. Or, place 56° 45′ on S to 165 on N, then againſt Radius on S is 197,3 on N, the Length of CF required. Theſe are all the Varieties of Operation by the *ſingle or ſliding Gunter*.

99. But it muſt here be remarked, that the *artificial Line of Sines* is very defective even for practical Uſes; for half the firſt Degree is always loſt at the Beginning of the Line; and at the End of it, many Degrees are uſeleſs, viz. all from 80 to 90; and thoſe from 70 to

80 are of little Uſe on common Rules. In ſhort, unleſs your Gunter be 2, 3, or 4 Feet long, the Degrees beyond 60 will hardly ſuffice for common Practice.

100. Since artificial Sines on Inſtruments are ſo deficient, we find the *natural Sines* and *Tangents* upon the SECTOR to ſupply, in ſome Meaſure, that Defect. But they are here laid down as they lie in the *Orthographic Projection*, and therefore are far from being ſo uſeful, as they are capable of being rendered in Practice, as we ſhall be convinced of by only inſpecting a SECTOR of 6 Inches Radius, where it will appear, that even in this Caſe the Degrees from 60 to 70, are of very little Uſe; and all beyond, are of none at all.

101. But though the *Line of Natural Sines* be ſo defective on the SECTOR, it is otherwiſe with the *Tangents*; they, inſtead of diminiſhing, increaſe as they proceed; ſo that it is neceſſary to have two Lines of natural Tangents, viz. one for *all below* 45°, and another for *all above*. Hence ſuch Analogies as conſiſt of Tangents, are wrought very advantageouſly upon the SECTOR. Note, the *Line of equal Parts* is upon the Sector called the LINE of LINES, and is marked with L at the End.

102. The Operation of the foregoing Example on the Sector, is in the following Manner: To find the Side GF from the Data (in 75) by the *Lines of Sines* S, S. Set one Foot of the Compaſſes in the Center of the Joint, and extend the other to 165 in the Line of Lines L; then with this Opening, ſet one Foot in 56° 45′ in the Line of Sines S, and open the Sector till the other Foot falls exactly on the ſame Diviſion 56° 45′ on the Line S upon the other Leg. The Sector remaining thus, take the parallel Diſtance from 33° 15′

to 33° 15′, in the Lines S, S, and ſet on Foot in the Center of the Sector, the other will fall upon 108,2 in the Line L, for the Side FG required.

103. On the *Lines of Tangents* T, T, the ſame Analogy is wrought thus; take 165 from the Line of Lines L, and opening the Sector apply it parallel-wiſe from 45° to 45, in the Lines T, T; then take the parallel Diſtance between 33° 15′ in each Line T, and apply it upon the Line L, and it will reach from the Beginning to 108,2, as before.

104. To find CF the Hypothenuſe, we muſt uſe the Sines S, S, thus: Take from the Line L the Number 165, and apply it parallel-wiſe between 56° 45′ upon each Line S; then take the parallel Diſtance of 90 and 90 on the ſame Lines, and apply it to the Line of Lines L, and it will reach from the Center to 197,3, the Length of CF required.

105. On Account of this Deficiency of the *Line* of SINES, both on the GUNTER and on the SECTOR, I have protracted the Sines as they ſtand in the natural Canon, and have thereby rendered this new LINE of SINES capable of all the Advantages it can poſſibly have, or indeed that the Artiſt can expect; for it has by this Means juſt the contrary Property to what it has on the Gunter or Sector, viz. the Diviſions continually encreaſe from the Beginning, and at the End they are very large; ſo that you have the moſt perfect Solution of ſuch Caſes of a Triangle here, as are altogether impracticable with this Line in any of the uſual Forms, as will be evident by the following Example.

106. Let the *Sun's Declination be* 22° 23′, *to find his Longitude.* The Obliquity of the Ecliptic is 23° 30′, therefore in the right-angled Spherical Triangle PZE (Fig. 3,)

(Fig. 3.) there is given the Angle E = 23° 30′, and the oppoſite Side B = 22° 23′, to find the Hypothenuſe C, by Variety II. *s*E : *s*B : : Radius : *s*C. Now on the Rule againſt 23° 30′ is the Sine 399, and againſt 22° 23′ is the Sine 381; therefore ſay, As 399 : 381 : : 1000 : 955 = Sine of C, againſt which in the Line of Degrees is 72° 38′, the Anſwer required.

107. Here it is evident, that ſince the Sine of 72° 38′ is required, it would be in vain to ſeek it on the artificial *Gunter*, or Line of *Sines* on the *Sector*, and yet by this new Conſtruction of the Line of Sines on the *Univerſal* SLIDING-RULE, the nearer you approach to 90°, the more perfect will the Solution be.

108. Again, *Suppoſe the* SUN *in* 10° 30′ *of Aries, what is his right Aſcenſion?* Here E = 23° 30′ and its Complement = 66° 30′; therefore by Caſe I. of Variety III. (41) ſay, As Radius : *cs*E : : *t*C : *t*A. In the Line of Sines, againſt 66° 30′ is 917; and againſt 10° 30′ is 185 in the Line of *Tangents*; therefore it will be, As 1000 : 917 : : 185 : 170 = C; then againſt 170 in the *Tangents* is 9° 39′ in the *Line of Degrees*, for the right Aſcenſion required. Note, theſe Analogies are wrought in the uſual Manner by the ſame SLIDING-RULE; as may be ſeen at large in my ſmall *Tract of the Deſcription and Uſe thereof, lately publiſhed.*

PATENT OFFICE

CHAP. X.

A General ENUMERATION of all *Aſtronomical*, *Nautical*, and *Geographical* PROBLEMS in *Spherical* GEOMETRY, illuſtrated by a DIAGRAM.

109. IT would be endleſs to recount all the Problems in the various Mathematical Sciences, to which the Application of the foregoing Trigonometrical Solutions is conſtantly made; it muſt ſuffice to point out ſome of the principal, with the Triangles from whence they ariſe, formed by the different Circles of the Sphere.

110. In each right-angled Spherical Triangle ſuppoſing only four Parts concerned, viz. the three Sides A, B, C, and the Angle E (Fig. 3.) there will be ſix Varieties of *Data*, viz. A, B; A, C; A, E; B, C; B, E; and C, E; and each of theſe finds two unknown Parts, and therefore contains two Problems; ſo that every ſuch Triangle furniſhes Solutions to 12 different and uſeful Problems. But ſometimes the Angle P comes in, and then 20 Problems will be ſolved by 10 Varieties of *Data*.

111. In an oblique ſpherical Triangle, there are three Sides and two Angles always concerned, viz. A, B, C, Z, P, (Fig. 4, 5.) and ſometimes the Angle E: and ſince the *Data* here conſiſt of three Parts, there will be 10 Varieties; and if only *two Parts* are ſought, there will be 20 Problems ariſing from every ſuch Triangle; and many more when the Angle E enters the Queſtion; in ſhort, 3 Quantities in 6 give 20 Varieties

in the *Data*, and 60 Problems reſult from thence, as is evident from the DOCTRINE of COMBINATIONS.

112. To give an Idea of the rich Variety of uſeful Problems, the *ſingle Diagram* will afford (Fig. 18.) let C be the *Center* of the Sphere; its *Axis* NS; North and South *Poles* N and S; ÆQ the *Equator*; EL the *Ecliptic*; EPLS the *Meridian*; HO the *Horizon*; ZCD the *Prime Vertical*; NBS an *Hour-Circle*, and ZAD a *Vertical Circle* paſſing through the *Place of the Sun* ⊙; PT the Axis of the Ecliptic; PRT a *Circle of Longitude*, and NRS a *Circle of Declination* paſſing through a *Star* at R. The *Tropic* of *Cancer* E♋, and of *Capricorn* ♑L.

113. By the Interſections of theſe Circles, a Variety of Triangles will be formed of both Sorts; ſome of the principal of which here follow:

114. I. The right-angled Triangle ⊙BC.

In which BC is the Sun's *Right Aſcenſion*.
C⊙ is his *Longitude* from the Equinox C.
B⊙ is his *Declination*.
BC⊙ is the *Obliquity of the Ecliptic*.
B⊙C the Angle of *Inclination* of the Meridian and Ecliptic.

115. II. The right-angled Triangle A⊙C.

In which C⊙ is the Sun's *Longitude*.
A⊙ his *Altitude* above the Horizon.
AC his Amplitude from C.
AC⊙ the Sum of the Co-Lat. and Obliq. of the Ecliptic.

116. III. The right-angled Triangle VCG; the Place of the Sun being at G in the Tropic of Cancer, and the Prime Vertical of Eaſt and Weſt.

Then VC is the *Hour from Six.*
VG the Sun's *greateſt Declination* 23° 28′.
CG his *Altitude when Eaſt or Weſt.*
VCG the Latitude of the Place Z.

117. IV. The right-angled Triangle ICM.

In which CI is the Sun's *Declination.*
IM his Altitude preciſely *at Six* o'Clock.
CM his *Co-Azimuth* at that Time.
ICM the Latitude of the Place.

118. V. The right-angled Triangle CWK, when the *Sun is in the Horizon* at K.

In which WK is the Sun's *Declination.*
CW is the *Aſcenſional Difference.*
CK his *Amplitude* from C, *Eaſt* or *Weſt.*
WCK the *Co-Latitude* of the Place.

119. VI. The right-angled Triangle NKO.

In which NO is the *Latitude* of the Place.
NK the *Co-Declination.*
KO the *Azimuth.*
KNO the *Hour-Angle* from Midnight.

120. And with regard to *oblique Triangles*, there are three of principal Note, viz. (1) NZ⊙, which reſpects the Pole (N) of the World, and the Sun or Star at ⊙; (2) PNR, which reſpects the Pole of the Ecliptic P, and a Star at R; and (3) ⊙ZR, which regards the Altitudes, Diſtances, and Azimuth of the Sun, Moon, or Star at ⊙ and R.

121. I. The Oblique Triangle NZ⊙.

In which ⊙Z is the Co-Altitude of the Sun.
⊙N is the Co-Declination.

In which ZN is the Co-Latitude of the Place.
ZN⊙ is the Hour Angle from Noon.
⊙ZN is the Azimuth from the North.

Note, The Triangle RZN anſwers the ſame Ends for a Star at R.

122. II. The Oblique Triangle PNR.

In which RN is the *Co-Declination* of the Star.
RP is its *Co-Latitude.*
NP is the *Obliquity of the Ecliptic.*
NPR the *Longitude* of the Star from E.
RNP the *Hour* from Midnight.

123. III. In the Oblique Triangle ⊙ZR.

In which ⊙Z is the *Co-Altitude* of the Sun.
RZ is the *Co-Altitude* of the Moon, or Star.
⊙R the *Diſtance* of the Sun and Moon, or Star.
⊙ZR the *Difference* of their *Azimuths.*

124. In theſe few Triangles, it is evident, there are upwards of 100 Problems contained, moſt of which daily occur in the practical Parts of ASTRONOMY, DIALLING, NAVIGATION, and GEOGRAPHY; for the Points ⊙, Z, N, &c. may as well be conceived on the Spherical Surface of the OCEAN or LAND, as in the HEAVENS. Thus ſuppoſe a Ship ſails from a Port Z to another X, then in the Triangle ZNX we have

ZN the *Co-Latitude* of the Port Z.
NX the *Co-Latitude* of the Port X.
ZX is the *Way of the Ship*, or *Diſtance* ſailed.
NZX is the *Rhumb*, or *Courſe* of the Ship.
ZNX is the *Difference* of *Longitude* of Z and X.

125. In this one Triangle, therefore, are contained all the Caſes of *Orthodromics* or *great Circle* SAILING. But Z and X may alſo repreſent *two* CITIES on the *terreſtrial* GLOBE; then will all the *Geographical Problems* relative to their *Latitude*, *Longitude*, *Diſtance*, *Poſition*, or *Bearing* from each other, admit of Solutions, in the uſual manner, by the Table of Proportions in Art. 52.

126. I may obſerve, in the laſt Place, that many of theſe Problems are of a general Nature, and contain many particular ones of moment; thus for Example, in the Triangle ☉ZN, the Angle at N is the Hour from Noon, when the Sun ☉ has the Altitude ☉A above the Horizon; but ſuppoſing the Sun in the Horizon at A, then the Angle ANZ will be the Time of the *Sun's riſing*; whence the LENGTH of DAY and NIGHT, and the SEASON of the YEAR will be known. Again, ſuppoſe the Sun were 18° below the Horizon at Y, then the ſaid Angle at N would be the Time when the *Day breaks*, or the TWILIGHT begins and ends.

CHAP. XI.

The *Fluxionary* PRINCIPLES of *Plain* TRIGONOMETRY explained, and applied to Practice.

127. ON the Center A, and with the Radius AC, deſcribe the Arch of a Circle BC, terminated by the Radii AB, AC. Let the Sine thereof be CD, and Co-ſine AD; and let the Arch BC be changed into the Arch BE, the Sine CD into the Sine

EF, and the Co-ſine AD into the Co-ſine AF; and draw CG parallel to AB, meeting the Sine EF in G. Alſo in B erect the perpendicular BI, which let the Radii AC, AE, continued out, cut in the Points H and I. And through H deſcribe the ſmall Arch HK with the Radius AH. (Fig. 19.)

128. Then put the *Arch* $BC = z$, and its Fluxion $CE = \dot{z}$; the *Sine* $CD = s$, and its Fluxion $EG = \dot{s}$; the *Tangent* $BH = t$, and its Fluxion $HI = \dot{t}$; the *Secant* $AH = ſ$, and its Fluxion $IK = \dot{ſ}$. Then, as the Angle HAI is indefinitely ſmall, the Arches CE and HK may be eſteemed *right Lines*, without any ſenſible Error.

129. Then by the ſimilar Triangles CEG and CAD, we have $CE : EG :: \dot{z} : \dot{s} :: CA : AD ::$ Radius : Co-ſine of the Angle BAC.

130. Again, by the ſimilar Triangles ACE and AHK, as alſo the Triangles ABH and HKI, we have $CE : HK :: AC : AH$, and $HK : HI :: AB : AH$; and therefore (*ex æquo*) it will be $CE : HI :: \dot{z} : \dot{t} :: AB^2 : AH^2$, that is, $\dot{z} : \dot{t} ::$ *the Square of Radius to the Square of the Secant of the Angle* A.

131. Laſtly, by the ſame ſimilar Triangles, we have $CE : HK :: AC : AH$, and $HK : IK :: AB : BH$; therefore (*ex æquo*) it will be $CE : IK :: AB^2 : BH \times AH$; that is, $\dot{z} : \dot{ſ} ::$ *Radius ſquare* : *the Rectangle under the Tangent and Secant.*

132. Let the oblique Triangle ABC (Fig. 20) be changed into the Triangle ABD, while any Angle B, and one adjacent Side AB, continue the ſame; make $AE = AC$, and join CE; then when the Angle CAD is indefinitely ſmall, we ſhall have (by the Triangle CDE, right angled at E) as $CD : DE :: R : s\,DCE$; that is,

the Fluxion of the other adjacent Side CB *is to the Fluxion of the oppoſite Side* AC, *as Radius is to the Co-ſine of the Angle* C, *which is oppoſite to the permanent Side* AB.

133. With a Radius AF (=100000) deſcribe the fluxionary Arch FG; then we have FG : CE : : R : AC; and alſo CE : DE : : *t*D (or C) : R; and from both theſe Analogies compounded, we have FG : DE : : *t*C : AC; that is, *the Fluxion of the Angle* A *is to the Fluxion of the Side* AC, *oppoſite to the conſtant Angle* B, *as the Tangent of* C *to the Side* AC.

134. Further, in the ſame Conſtruction, we have FG : CE : : R : AC; and CE : CD : : *s*C (or D) : R : and from both Analogies compounded, we have FG : CD : : *s*C (or D) : AC; that is, *the Fluxion of the Angle* A (*or* C) *is to the Fluxion of the other Side* BC *adjacent to the permanent Angle* B, *as the Sine of the Angle* C *oppoſite to the conſtant Side* AB, *is to the Side* AC, *oppoſite to the permanent Angle* B.

135. If the *oblique Triangle* ABC were changed into the *right-angled Triangle* A*B*C, or the Angle at B made right at *B*, the ſame Proportions amongſt the Fluxions of the reſpective Sides and Angles will continue as before, with the conſtant Side A*B*, and Angle *B*.

136. If the ſame Conſtruction were made at the Angle A; and the Side BC, or *B*C, with the Angle B, or *B* conſtant, then by the ſame Reaſoning we ſhould have, *the Fluxion of the Angle* A (*or* C) *to the Fluxion of the Side* AB, *or* A*B*, *as the Sine of the Angle at* A *to the Side* AC.

137. To ſhew the Uſe of theſe fluxionary Principles, let A*B*=H be the Height of an Object to be meaſured, the Queſtion is, *at what Diſtance* B*C you are to take your Station, for the Angle* AC*B to be liable to the leaſt*

Error poſſible. This Problem is ſolved as follows; Continue AB to (a), and make $Ba=BA$, and join aC; and let $\dot{H}$ be the Fluxion of AB, and $\dot{C}$ that of the Angle ACB. Then we have (as above) As $\dot{H} : \dot{C} :: AC : sA$. But the Angle $BAC=a$, and the Angle $ACa=2ACB$; therefore $AC : sA$ (or $s.a$) $::$ Aa ($=2AB$:) $sACa$ ($=s2ACB$) that is, $\dot{H} : \dot{C} :: 2AB : s.2ACB$; therefore $\dot{H} : AB :: 2\dot{C} : s.2ACB$.

138. By this Analogy, it is evident, the Error $\dot{H}$ in the Height of the Object AB, has the ſame Ratio to that Height, as double the Error $\dot{C}$ of the Angle of Altitude, has to the Sine of double that Angle; conſequently this Ratio will be the leaſt of all, when the Sine of $2\dot{C}$ is greateſt, viz. when it becomes *Radius*, or when the Angle $ACB=45$ Degrees.

139. This Example of the Uſe of *fluxionary* TRIGONOMETRY is of ſuch Importance as to deſerve a particular Illuſtration in Numbers. To this Purpoſe, let BH (Fig. 19.) be the given Object, and make AB Radius$=1000000$ equal Parts; then BAH will be the *true Angle* of Altitude, and BAI the *erroneous Angle* given by the Quadrant. Let EC, the Error of the Angle, be *one Minute*, or $\dot{C}=1'$; then in the trigonometrical Canon, you find the Arch of $1'$ to be 291 of the equal Parts of Radius; the double whereof is $2\dot{C}=582$; therefore the Ratio of $\frac{582}{1000000}=\frac{1}{1719}=\frac{HI}{BH}$; that is, the Error HI is but $\frac{1}{1719}$ Part of the Height BH of the Object, when the Angle BAH$=45°$, and is then a *Minimum*.

140. If the Angle at A be greater or leſs than 45°, then (other Things remaining the ſame) *the Error* HI *will be increaſed in the Ratio of Radius to the Sine of double the Angle* A. If the Error CE of the Angle A, be greater or leſs, then (other Parts remaining the ſame) the Error HI will increaſe or decreaſe in the ſame Ratio.

C H A P. XII.

The Fluxionary PRINCIPLES of Spherical TRIGONOMETRY explained, and applied in the SOLUTION of ſome very important PROBLEMS.

141. LET ABC be an oblique ſpherical Triangle (Fig. 21.) and let the Side AC move into the Poſition A*c*, deſcribing the Parallel C*c*; alſo let the Side BC in the ſame Time, move into B*cd*, deſcribing the ſmall Arch of a Parallel C*d*. Again, let the three Sides of the Triangle be denoted by D, E, F; and the 3 Angles by A, B, C.

142. Now ſuppoſe two Sides AB and AC *conſtant*, and all the other Parts variable; then becauſe the fluxionary Triangle CA*c* is right-angled at (*c*) we have R : *s*.E : : *s*.CA*c* : *s*.C*c*; and becauſe the Sine of a very ſmall Arch is nearly equal to the Arch itſelf; therefore it will be $R : s.E :: CAc \ (= \dot{A}) : Cc = \frac{sE}{R} \times \dot{A}$.

In the ſame Manner, $R : s.F : : \dot{B} : Cd = \frac{s.F}{R} \times \dot{B}$.

143. Becauſe $ACc=dCB=$right Angle; therefore $ACB=dCc$, and the Angle at (d) being a right one, we have $R : s.dCc\ (=ACB) :: cC : cd :: \frac{sE}{R}\times\dot{A} : \dot{F} = \frac{s.E\times s.C}{R^2}\times\dot{A}$.

144. But $s.C : s.D :: sB : sE$ (by 47.) therefore $sE\times sC=sD\times sB$, and ſo $\dot{F}=\frac{sD\times sB}{R^2}\times\dot{A}$.

145. Again $R : cs.\ dCc\ (=C) :: Cc : Cd :: \frac{sE}{R}\times\dot{A} : \frac{sE\times csC}{R^2}\times\dot{A}=\frac{sF}{R}\times\dot{B}$; therefore we have $\dot{B}=\frac{sE\times cs.C}{R\times sF}\times\dot{A}$. (142.)

146. Laſtly; it is $ct.\ cCd\ (=C) : R :: Cd : cd :: \frac{sF}{R}\times\dot{B} : \dot{F}=\frac{sF}{ct.C}\times\dot{B}$.

147. In the ſame Manner it is ſhewn, (by making the ſame Conſtruction at the Angle B), that $\dot{C}=\frac{sD\times csB}{R\times sF}\times\dot{A}$; and $\dot{F}=\frac{sF}{ct.B}\times\dot{C}$.

148. Let $ſ=$Secant, $cſ=$Co-ſecant of any Arch DF (Fig. 1.); then we have $R^2=sDF\times cſDF$. (19.) and conſequently $R^2=\frac{sD\times sB}{\dot{F}}\times\dot{A}$ (144.) $=sD\times cſD$; whence we ſhall have the following fluxionary Analogies, viz.

149. I $\dot{A} : \dot{F} :: cſD : sB$.
II $\dot{A} : \dot{B} :: R\times sF : sE\times csC$ (145).
III $\dot{A} : \dot{C} :: R\times sF : sD\times csB$. (147).
IV $\dot{B} : \dot{F} :: ct.C : sF$. (146).
V $\dot{C} : \dot{F} :: ctB : sF$. (147).
VI $\dot{B} : \dot{C} :: ctC : ct\,B :: tB : tC$. (146, 147)

150. Theſe Analogies will ſuffice for our preſent Purpoſe, which is to ſhow the great Uſe of the *Fluxionary Calculus* in the Solution of many Problems in *Spherical* ASTRONOMY, which though of the utmoſt Importance, are otherwiſe either quite unattainable, or elſe require the moſt operoſe and difficult Proceſs. The firſt Example of this Kind ſhall be the following, viz.

151. *To find on what Azimuth or vertical Circle* ZR. *you can moſt advantageouſly take the Altitude* RE, *of any celeſtial Object* E. (See Fig. 2.)

In the Triangle PZE, the Side PZ the Co-latitude, and PE the Co-declination, are conſtant; and the Side ZE the Co-altitude, with the Angles P, Z, E, variable; the Relation of the Fluxion of the *Hour-Angle* P to that of the oppoſite Side ZF, is given in the firſt of the foregoing Analogies, viz. $\dot{A} : \dot{F} :: cſD : sB$.

152. That is, in the preſent Caſe, (putting $L =$ *Latitude*, $D =$ *Declination*, $A =$ *Altitude*, $H =$ *Hour-Angle*, and $Z =$ Azimuth,) $\dot{H} : \dot{A} :: ſL : s.Z.$ viz. *the Fluxion of the Hour-Angle is to the Fluxion of Altitude, as the Secant of the Latitude is to the Sine of the Azimuth.* Now becauſe $\dot{H} = \frac{\dot{A} \times ſL}{sZ}$, it is evident, the Altitude of the Object E does not affect the Variation or Fluxion of the Hour-Angle at all, as the Quantity A is not concerned in the Equation, but only its Fluxion $\dot{A}$.

153. Again, in the ſame Latitude L, we have $\dot{H} = \frac{\dot{A}}{sZ}$; therefore $\dot{H}$ will be leſs as the Sine of Z encreaſes; and conſequently will be *leaſt of all* when the Luminary is upon the *Prime Vertical* ZCz where $sZ = R$; which ſhews, that in that Situation a judici-

ous Obſerver will take its Altitude, when the *Apparent Time* is to be deduced from it.

154. If we ſuppoſe $ſL = sZ =$ Radius, it brings the Obſerver to the EQUATOR, and places the Object alſo in the Equator; and in that Caſe we have $\dot{H} = \dot{A}$, the *leaſt it poſſibly can be*. Here let the Error in the Angle of Altitude $\dot{A} = 1$ Minute, then $\dot{H} = 1'$ alſo; but 1′ in the *Hour-Angle*, or of *Motion* in the Equator, is performed in 4″ of Time: By the Mean of ſeveral Altitudes the Error $\dot{A}$ may be diminiſhed, ſuppoſe to $\frac{1}{4}$ of a Minute, then that will give (under the *Equinoctial*) an Error $\dot{H}$ of 1″ of Time only.

155. By the above Theorem (152) the *Error in Time* is found for any Latitude, when the Star is upon the Prime Vertical, and the Error in Altitude 1′; thus in Lat. 45° the ſaid Error will be 5″ 20‴; in Lat. 50° it will be 6″. 13‴; in Lat. 55°, the Error will be 7″ nearly.

156. The next Example I ſhall give is pregnant with the cleareſt *Principles of Fluxionary* TRIGONOMETRY, and contains, in ſhort, the very *Rationale* of it. Let P be the POLE of the World, PE the *Equinoctial* COLURE, PI the *Solſtitial* COLURE, EI the EQUINOCTIAL, EH the ECLIPTIC, and PD a MERIDIAN; all in *quadrantal* Arches. (Fig. 22.) Alſo draw PB, P*d*, and PG to conſtitute *fluxionary quadrantal Triangles*.

157. Then in the Triangle DP*d*, we have the *s*PD : *s*PC : : D*d* : C*e* (ſuppoſing C*e* parallel to D*d*.) that is, $R : sPC :: Dd\ (= \dot{a}) : Ce = \frac{sPC \times \dot{a}}{R} = \frac{csDC \times \dot{a}}{R}$.

158. Again, the ſmall fluxionary Triangle c Ce gives this Analogy; tC (=c) : R : : Ce : ce : : $\frac{cs\text{DC}\times\dot{a}}{\text{R}}$: $\frac{cs\text{DC}}{t\text{C}}\times\dot{a}=ce=\dot{x}$, the Fluxion of DC.

159. In the ſame Triangle cCe, as sC : R : : Ce : Cc : : $\frac{cs\text{DC}}{\text{R}}\times\dot{a}$: $\frac{cs.\text{DC}}{s.\text{C}}\times\dot{a}=\text{C}c=\dot{z}$, the Fluxion of the Arch EC. Whence $\dot{a}$: $\dot{z}$: : sC : csDC, under the Meridian PD.

160. At the *Equinox* E, the ſmall Triangle ABE gives EB : EA : : (csAEB : R : :) $\dot{a}$: $\dot{z}$. And the Triangle GPI at the *Solſtice*, gives GI : FH : : R : csIH (=AEB) : : $\dot{a}$: $\dot{y}$.

161. Hence $\dot{a}$: $\dot{z}$: : $\dot{y}$: $\dot{a}$, and ſo $\dot{z}\dot{y}=\dot{a}^2$, a *conſtant Quantity*; whence it appears, that as the SPHERE of the HEAVENS revolves under the MERIDIAN, the Fluxion or Velocity of Motion in the Equator EI, is every where the *ſame*, viz. $\dot{a}$; that of the Ecliptic is variable from $\dot{z}$ at E to $\dot{y}$ at H: And that $\dot{z}$ is greater than $\dot{a}$ in the ſame Ratio that $\dot{a}$ is greater than $\dot{y}$.

162. There is, therefore, ſome intermediate Point C, where the Velocity of the Ecliptic is equal to that of the Equator, or Cc=Dd. To determine this, we have every where $\dot{a}$: $\dot{z}$: : sC : csDC (159) And by the 2d Analogy of Variety VI (in 41) we have sC : csE : : R : csCD; therefore $s\text{C}=\frac{cs\text{DC}}{\dot{z}}\times\dot{a}=\frac{cs\text{E}\times\text{R}}{cs\text{CD}}$; whence $cs\text{DC}^2 : cs\text{E}\times\text{R} : : \dot{z} : \dot{a}$.

163. Now putting $\dot{z}=\dot{a}$ (or Cc=Dd,) we have $cs\text{DC}^2=cs\text{E}\times\text{R}$; and conſequently, $cs\text{DC}=\sqrt{cs\text{E}\times\text{R}}$, The Value of E is 23° 28′, its complement 66° 32′; whence by Computation we find the *Declination* DC=

16° 43′; the *Longitude* EC=46° 15′; the *Right-Aſcenſion* ED=43° 47′; and therefore EC−ED=D*d* =2° 28′, a *Maximum.*

164. This Arch D*d* is, by *Aſtronomers*, called the EQUATION of TIME, the Quantity of which, for every Degree of the Ecliptic, is computed and turned into Time, which *added to*, or *deducted from* the *Solar* TIME by the DIAL, gives the *Mean* TIME, for a CLOCK or WATCH. But for a larger and compleat Explication of this Doctrine, ſee my TREATISE of CLOCK-WORK, Chap. XXV. &c.

165. From theſe Examples it is plain, that when the Fluxions of two flowing Quantities are equal, the *Difference of thoſe Quantities* is a *Maximum* or *Minimum.*

166. A *Third Example* ſhall be to find the Relation of the Angles P*e*Z, and P*E*Z which every Hour-Circle P*e*, P*E* makes with the Correſpondent Vertical, *that the Time which a Star takes in paſſing from one Almicantar to the other may be the leaſt poſſible.* (Fig. 2.)

Let (*e*) be the Place of the Star in the Almicantar L*e*, and *E* its Place in any other MN; then in the given Latitude PZ=L, and Altitude Z*e*=A, with the variable Declination *se*=D, we have the Relation of the Fluxion of the Hour-Angle ZP*e*=$\dot{\mathrm{H}}$, to the Fluxion of the Co-Declination P*e* thus expreſſed, $\dot{\mathrm{H}} : -\dot{\mathrm{D}} :: ct.e : cs\mathrm{D}$ (by 149) Therefore $\dot{\mathrm{H}} = ct.e \times \frac{-\dot{\mathrm{D}}}{cs\mathrm{D}}$.

167. Alſo in the Almicantar MN, the Place of the Star being *E*, we have the Hour-Angle ZP*E*=H, and becauſe *E* is in the ſame Parallel of Declination D*E*, we have $\dot{\mathrm{H}} = ct.E \times \frac{-\dot{\mathrm{D}}}{cs\mathrm{D}}$. Therefore $\dot{\mathrm{H}} : \dot{H} ::$

$ct.e : ct E$; but when $\dot{\mathrm{H}}=\dot{H}$, then the Angle $e=\mathrm{E}$; and conſequently the Difference *E*P*e* of the flowing Angles will then be a *Minimum*; (by 165) and the Star will in that Caſe paſs from *e* to *E* in the *leaſt Time* poſſible.

168. Hence it appears, that there is a certain Declination which will cauſe the Angle E to be conſtantly the ſame while the Parallel DE*d* is deſcribed by the Star; and this Declination is found for any given Altitudes or Diſtances of *Almicantars* L*l*, or MN, on either Side the Horizon HR*h*, by Theorem I (at 28), which gives for the Point (*e*) this Equation $\frac{cs.\mathrm{D}\times cs.e}{\mathrm{R}}=\frac{\mathrm{R}\times s\mathrm{L}-s\mathrm{A}\times s\mathrm{D}}{cs.\mathrm{A}}$; in the ſame Manner for any other Point *E*, we have $\frac{cs\mathrm{D}\times cs.E}{\mathrm{R}}=\frac{\mathrm{R}\times s\mathrm{L}+s.\mathrm{a}\times s\mathrm{D}}{cs.\mathrm{a}}$. But becauſe the Angles *E* and *e* are equal, the firſt Parts of the above Equations are equal; therefore alſo the ſecond are ſo, viz. $\frac{\mathrm{R}\times s\mathrm{L}-s\mathrm{A}\times s\mathrm{D}}{cs.\mathrm{A}}=\frac{\mathrm{R}\times s\mathrm{L}+s.\mathrm{a}\times s.\mathrm{D}}{cs.\mathrm{a}}$; from whence we get the following Analogy; $s.\mathrm{a}\times cs.\mathrm{A}+s\mathrm{A}\times cs.\mathrm{a} : \mathrm{R}\times\overline{cs.\mathrm{a}-cs.\mathrm{A}} :: s\mathrm{L} : s.\mathrm{D}$, the *Declination* ſought.

169. In the above Equations, A is the Altitude above the Horizon, and *a* the Depreſſion below it; and becauſe *EZ* exceeds 90°, the Sign + was placed before $s.\mathrm{a}$ in the ſecond Equation for the Point *E*.

170. If one of thoſe *Almicantars* as L*l*, be ſuppoſed to coincide with the Horizon, then $\mathrm{Z}e=\mathrm{Z}t$, and $s.\mathrm{A}=o$, and $cs\mathrm{A}=\mathrm{R}$, in the above Analogy (168), which then becomes $s.\mathrm{a}\times\mathrm{R} : cs\mathrm{a}\times\overline{\mathrm{R}-\mathrm{R}^2} :: s.\mathrm{a} : cs.\mathrm{a}-\mathrm{R} :: s\mathrm{L} : s.\mathrm{D}$.

171. Hence we have an eaſy Solution to that famous and difficult Problem *of finding the Day when the* TWILIGHT *is ſhorteſt of all in the Year.* It is agreed by Aſtronomers that the Twilight begins or ends when the Sun is in the Parallel MN of 18 Degrees below the Horizon; therefore in the Latitude of LONDON, where $L=51^\circ.\ 31'$, we have $sa : cs.a-R :: sL : sD=7^\circ\ 7'$ nearly; which Declination is South, or of a contrary Denomination to the Latitude, as is indicated by the negative Term $csa-R$. Hence the Day of the *ſhorteſt Twilight* is March 2d.

172. A more ſimple and elegant Equation than the above is obtained as follows. Let the Arch $DF=a$ (Fig. 1.) then $R=CD$, $FG=sa$, $CG=csa$, and $GD=v=$the verſed Sine of a. Now 'tis evident $R-CG=R-csa=v$, and therefore $csa-R=-v$. Again, join DF and FA, then is the Angle $FAD=\frac{1}{2}FCD=\frac{1}{2}a$, and making AF Radius, $FD=$Tangent of $\frac{1}{2}a$, then from the ſimilar Triangles FGD, FAD, we have $FG : GD :: AF : FD$, that is, $s.a : v :: R : t.\frac{1}{2}a$; and conſequently $s.a : -v\ (=csa-R) :: R : -t\frac{1}{2}a$. Therefore laſtly, $R : -t\frac{1}{2}a :: sL : sD$, or *Radius is to the* Tangent of 9° : : Sine of 51°. 30′ : Sine of 7°. 7′. the *Sun's Declination*, as above.

CHAP. XIII.

A New and Eaſy METHOD of ſolving all the CASES of Oblique Spherical TRIANGLES by *Six right-angled Subſidiary* TRIANGLES.

173. BESIDES the Methods of ſolving oblique Spherical Triangles already taught, there is *one more*, which I cannot ſuffer myſelf to omit, as it is in itſelf very ſcientific, leſs ambiguous than the uſual Method of a Perpendicular (43, &c.) and in every Analogy, *Radius* is the *firſt Term*; conſequently on all theſe Accounts, this Expedient muſt be very acceptable to the *young Trigonometer*, and ſtill the more ſo, as it is not to be found in any *Engliſh Treatiſe*, that I know of

174. This Artifice conſiſts in the Uſe of *Six Subſidiary right-angled Triangles*, two of which will always contain four of the *Six Parts* in the *Original Triangle*, or that propoſed for Solution; an Idea of this Matter will be eaſy to conceive from Fig. (23,24) in which ABC is the Triangle propoſed, one of whoſe Sides as AB, is continued out to an entire Circle. Then on the Points A and B, as Poles, deſcribe the two Semicircles EKQ, and HKR, which will biſect each other in the Point K; and in an Angle equal to the Arch AB, the Diſtance of their Poles. Alſo continue the Sides AC, BC, to Semi-circles ACM, BCN, which will be ſeverally biſected by the Semi-circles EKQ and HKR in the Points D and T; but unequally divided

by them in the Points G, I. Laſtly, join the Points CK by the Arch of a great Circle.

175. Then you will have formed *Six Subſidiary* Triangles, viz Three GDK, CDK, CID, right angled at D; and three others, CTK, ITK, CGT, right angled at T; the Sides and Angles of all which are determined by thoſe of the Triangle ABC. Thus AB is the Meaſure of the Angle DKG, or of its Supplement; AC and BC are the Complements of DC and CT; and the Angles ABC, BAC, are Complements of the Sides TK and KD.

176. By Means of two of theſe *ſubſidiary Triangles*, therefore, we are enabled to ſolve all the Caſes and Varieties of oblique Spherical Triangles, as will be evident by the following Examples, in which the Firſt Term of every Analogy is Radius.

EXAMPLE I.

177. *Given two Angles* A, *and* B, *and the included Side* AB, *to find one of the other Sides, as* BC.

Here the two ſubſidiary Triangles DKG, and CGT, furniſh the three following Analogies for a Solution, viz.

Analogy I. Radius : *s*A : : *s*AB : *cs*DGK. (by Art. 38.)

Analogy II. Radius : *cs*AB : : *t*A : *ct*GK. (by 36.)

Then GK − TK (the Complement of B) gives GT; and then, the Triangle CGT gives

Analogy III. Radius : *s*GT : : *t*DGK : *ct*·BC. (by 36)

EXAMPLE II.

178. *From the ſame Data, to find the third Angle* C.

The ſame ſubſidiary Triangles ſerve here; and the two firſt Analogies are the ſame, and give GT, as before.

Then, ANALOGY III. Radius : *s*GT : : *s*DGK : *cs*BC. (by 38.)

EXAMPLE III.

179. *Two Sides* AB, AC, *and the contained Angle* A, *being given, to find one of the other Angles, as* B.

Here the two ſubſidiary Triangles CDK, CKT, give

ANALOGY I. Radius : *s*A : : *s*AC : *cs*CK.
ANALOGY II. Radius : *cs*A : : *t*AC : *ct*CKD.

Then CKD ± DKT = CKT. Whence

ANALOGY III. Radius : *cs*CKT : : *t*CK : *ct*B.

EXAMPLE IV.

180. *By the ſame Data, to find the third Side* BC.

The ſame Subſidiary Triangles give this Analogy, Radius : *s*.CK : : *s*.CKT : *cs*BC.

Alſo by the Subſidiary Triangles DKG, CTG, we have

ANALOGY I. Radius : *s*A : : *s*AB : *cs*DGK.
ANALOGY II. Radius : *cs*A : : *t*AB : *t*GD.

Then GD ± DC = GC; whence

ANALOGY III. Radius : *s*GC : : *s*.DGK : *cs*BC.

The more extenſive Application of this *new and eaſy Method* of ſolving the Caſes of Oblique Triangles, will be left for the *young Trigonometer's* own *Praxis*.

PATENT OFFICE

CHAP. XIV.

The Method of ſolving oblique Spherical Triangles, and many other Problems, by the Sums and Differences of Sines, Co-Sines, and Tangents of two given Arches of a Circle.

181. THERE is another Invention for ſolving any Caſe of an *Oblique Spherical Triangle* by the foregoing new *Theorems*, without having any regard to *Natural Sines or Tangents* at all; performing every Thing by their *Logarithms only*. The Reaſon of which is explained by a Diagram, as follows (See Fig. 25.)

182. Let PE be an Arch of a Circle denoted by X; and PB another Arch=Y; let C be the Center, and PN the Diameter of the Circle; EO the Chord of 2EP; and draw BM and BH perpendicular to EO and NP. Alſo draw EB and BO, and pependicular to theſe, draw CR, CT, terminating the Tangent RBT to the Point B. Laſtly, draw MO, ME, CB; and AL perpendicular to CB. This Conſtruction well conſidered, the following Particulars will be evident.

183. (1.) The Arch $BO=X+Y$. (2.) $BE=X-Y$. (3.) $KO=ED+HB=s.X+s.Y$. (4.) $KE=ED-BH=s.X-s.Y$. (5.) $KM=HC+DC=cs.X+cs.Y$. (6.) $KB=csX-cs.Y$. (7.) $BT=t.\overline{\tfrac{1}{2}X+\tfrac{1}{2}Y}$. (8.) $BR=t.\overline{\tfrac{1}{2}X-\tfrac{1}{2}Y}$. (9.) $AL=s.\overline{\tfrac{1}{2}X+\tfrac{1}{2}Y}$. (10.) $CL=cs.\overline{\tfrac{1}{2}X+\tfrac{1}{2}Y}$. (11.) $BF=s.\overline{\tfrac{1}{2}X-\tfrac{1}{2}Y}$. (12.) $CF=cs.\overline{\tfrac{1}{2}X-\tfrac{1}{2}Y}$. (13.) $OM=2cs.\overline{\tfrac{1}{2}X-\tfrac{1}{2}Y}$. (14) $ME=2cs.\overline{\tfrac{1}{2}X+\tfrac{1}{2}Y}$.

184. Note, the Truth of the 13th. and 14th. Articles may not, perhaps, appear ſelf-evident at firſt Sight; but they are eaſily made out thus. The Arch $MO = MN + NO = Y + NO = Y + 180^\circ - X$. Therefore $\frac{1}{2}MO$ (the Chord) being the Sine of $\frac{1}{2}$ the Arch MO; it is plain $\frac{1}{2}MO = s.\overline{90^\circ + \frac{1}{2}Y - \frac{1}{2}X}$; conſequently $MO = 2s.\overline{90^\circ + \frac{1}{2}Y - \frac{1}{2}X}$, or $MO = 2cs.\overline{\frac{1}{2}X - \frac{1}{2}Y}$. In the ſame Manner the 14th Article is proved true. From this Preparation the following Theorems are derived.

THEOREM I.

185. $s.X + s.Y = 2s.\overline{\frac{1}{2}X \times \frac{1}{2}Y} \times cs.\overline{\frac{1}{2}X - {}^{1}Y}$.

For the Triangles MKO and CLA, right angled at K and L, are evidently ſimilar, the Angle at the Circumference M being equal to that at the Center C, as ſtanding upon twice the Arch BAO; therefore we ſhall have this Proportion, OK : OM : : AL : CA; that is, by ſubſtituting the Value of theſe Lines in *Sines* and *Co-ſines*, $s.X + sY : 2cs.\overline{\frac{1}{2}X - \frac{1}{2}Y} : : s.\overline{\frac{1}{2}X + \frac{1}{2}Y} : 1$; from whence by multiplying Extremes and Means, you have the Theorem as above ſtated.

THEOREM II.

186. $s.X - s.Y = 2cs\,\overline{\frac{1}{2}X + \frac{1}{2}Y} \times s.\overline{\frac{1}{2}X - \frac{1}{2}Y}$.

For the Triangles EKB and CLA being ſimilar (for the ſame Reaſons as before) give this Analogy EK : EB : : CL : CA; that is, by Subſtitution, $s.X - sY : 2s.\overline{\frac{1}{2}X - \frac{1}{2}Y} : : cs.\overline{\frac{1}{2}X + \frac{1}{2}Y} : 1$. which produces the Theorem above.

THEOREM

THEOREM III.

187. $cs.X + cs.Y = 2cs.\overline{\tfrac{1}{2}X + \tfrac{1}{2}Y} \times cs.\overline{{}_{2}X - \tfrac{1}{2}Y}$.

For the ſame right-angled Triangles MKO, CLA, give MK : MO : : CL : CA, that is, by ſubſtituting the Value of theſe Lines, $csX + cs.Y : 2cs.\overline{\tfrac{1}{2}X - \tfrac{1}{2}Y} : : cs.\overline{\tfrac{1}{2}X + \tfrac{1}{2}Y} : 1$; from whence the Theorem is derived.

THEOREM IV.

188. $csX - cs.Y = 2s.\overline{\tfrac{1}{2}X + \tfrac{1}{2}Y} \times s.\overline{\tfrac{1}{2}X - \tfrac{1}{2}Y}$.

For from the Triangles EKB, CLA, we have as before, BK : BE : : AL : AC, that is, $cs.X - cs.Y : 2s.\overline{\tfrac{1}{2}X - \tfrac{1}{2}Y} : : s.\overline{\tfrac{1}{2}X + \tfrac{1}{2}Y} : 1$. which gives the preſent Theorem.

THEOREM V.

189. $$\frac{sX + sY}{sX - sY} = \frac{t.\overline{\tfrac{1}{2}X + \tfrac{1}{2}Y}}{t.\overline{\tfrac{1}{2}X - \tfrac{1}{2}Y}}$$

For the firſt Part of the Equation is (by Theorems I. and II.) $= \dfrac{2s.\overline{\tfrac{1}{2}X + \tfrac{1}{2}Y} \times cs\overline{\tfrac{1}{2}X - \tfrac{1}{2}Y}}{2cs.\overline{\tfrac{1}{2}X + \tfrac{1}{2}Y} \times s.\overline{\tfrac{1}{2}X - \tfrac{1}{2}Y}}$. And becauſe $t = \dfrac{s}{cs}$, and alſo $\dfrac{cs}{s} = \dfrac{1}{t}$ (24.) therefore, ſubſtituting theſe *Tangents* inſtead of *Sines* and *Co-ſines*, this very Fraction becomes the ſecond Part of the Equation in the Theorem; whence the Theorem is evidently true.

190. In order to apply theſe five Theorems to the former, or rather to convert them into theſe Forms, it is needful only to conſider, that *Sines*, *Tangents*, &c. and their *Products*, are only certain Decimal Numbers below *Unity* (or Radius) 1,0000000000 whoſe Logarithm is 0,000000; therefore any two Numbers

propoſed, as 8635, and 8467 may be conſidered as the *Sines* of two Arches X, and Y; and then being put into the decimal Form of Sines 0,8635000000; and 0,8467000000, their Logarithms will be 9,936262, and 9.927730.

191. Now in the Canon of *Logarithmic Sines*, you find againſt

$$\left\{\begin{array}{lcl} 9{,}936262 & - & 59^\circ.\ 42'.\ 40''=X. \\ 9{,}927730 & - & 57.\ \ 51.\ \ 16=Y. \end{array}\right.$$

$$\begin{array}{lcl} \text{Half Sum is} & & 58.\ \ 46.\ \ 58=\tfrac{1}{2}X+\tfrac{1}{2}Y. \\ \text{Half Diff.} & - & 0.\ \ 55.\ \ 42=\tfrac{1}{2}X-\tfrac{1}{2}Y. \end{array}$$

Then by the ſecond Part of the Equation of Theorem I. we have the Logarithm of

$s.\overline{\tfrac{1}{2}X+\tfrac{1}{2}Y}$	- -	$58^\circ\ 46\ 58''$	9,932072
$cs.\overline{\tfrac{1}{2}X-\tfrac{1}{2}Y}$	- -	0 55 42	9,999914
2	- - -		0,301030
The Sum is the Logarithm of $s.X+s.Y$ viz 1,7102000000			0,233016

192. By this Operation it appears that the *Logarithm of the Sum* 17102 of two Numbers 8635 and 8467 may be found by a *Table of Logarithmic Sines alone*, independent of the uſual Method of *proportional Parts* in the Canon for common Numbers.

193. For another Example, let Theorem III. (Art. 30) be put into the Form of Theorem II. here; thus put the triple Product $tZ,\times csB\times csP=sX$, and $sP=sY$; then the Theorem becomes $\frac{sX-sY}{tZ\times sB}=ctC=$

$$\frac{2.cs.\overline{\tfrac{1}{2}X+\tfrac{1}{2}Y}\times s.\overline{\tfrac{1}{2}X-\tfrac{1}{2}Y}}{tZ,\times sB}.$$

194. This Example we ſhall illuſtrate by Numbers; thus let the Angle at Z=134° 38', or 45° 22; the

Angle P=27° 08′; and the Side B=41° 10; to find the Side C? Here then we have

*t*Z	45 22	10.005559	*t*Z	10,005559
*cs*B	48 50	9,876678	*s*B	9,818392
*cs*P	62 52	9,949331		
			Diviſor	9,823951
X =	42 44	9,831568		

Then X = 42° 44′
Y = 27 08

$\frac{1}{2}X+\frac{1}{2}Y$ = 34. 56	Coſine -	9,913718
$\frac{1}{2}X-\frac{1}{2}Y$ = 7. 48	Sine - -	9,132630
	2 - -	0,301030
	Sum - -	9,347378
Subduct *t*Z × *s*B	- - -	9,823951
There remains *ct*C = 18° 28′	- -	9,523427

Therefore the Side C = 71. 32, required.

195. Thus you transform any other Theorem, and find the required Parts of an Oblique Spherical Triangle without regard to letting fall a *Perpendicular*, or to *Natural Sines and Tangents* of Arches; but becauſe this may be deemed a difficult or troubleſome Method, (though a very ingenious one) I ſhall leave the farther Proſecution of it to the young Trigonometer's Exerciſe.

CHAP.

C H A P. XV.

An INVESTIGATION of SERIES for *computing* the VALUE of the PERIPHERY of a CIRCLE; and of the SINE, TANGENT, and SECANT of any given ARCH or ANGLE, in *equal* PARTS of RADIUS.

196. THE Radius of a Circle being ſuppoſed to conſiſt of 10000000000 equal Parts, we can, from the Principles of Trigonometry, very eaſily find the LENGTH of the CIRCUMFERENCE thereof; and that of a SINE, TANGENT, or SECANT of any propoſed Angle, expreſſed in the ſame equal Parts, in the Manner following, from the learned Dr. HALLEY.

197. Let A be the Center of a Circle (Fig. 19.) the Radius thereof AB$=r$; any given Arch BC$=z$; its Sine CD$=x$; Tangent BH$=t$; Secant AH$=ſ$; draw AI infinitely near to AH, and all the other Lines as in Art. 127. Then GE$=\dot{x}$, HI$=\dot{t}$, KI$=\dot{ſ}$, and CE$=\dot{z}$.

198. Then AH^2 $(=r^2+t^2)$: AB^2 (r^2) : : $\dot{t}$: $\dot{z}$ (Art. 130.) Therefore $\frac{r^2\dot{t}}{r^2+t^2}=\dot{z}=\dot{t}-\frac{t^2\dot{t}}{r^2}+\frac{t^4\dot{t}}{r^4}-\frac{t^6\dot{t}}{r^6}+\frac{t^8\dot{t}}{r^8}$, &c. The Fluent of which is $t-\frac{t^3}{3r^2}+\frac{t^5}{5r^4}-\frac{t^7}{7r}+\frac{t^9}{9r^8}$ &c.$=z$, or the Arch BC required.

199. If now we put AB$=r=1$, then the Arch BC$=z=t-\frac{1}{3}t^3+\frac{1}{5}t^5-\frac{1}{7}t^7$, &c. And if we ſuppoſe the Arch BC$=30$ Degrees, then the Sine CD$=\frac{1}{2}$AB$=\frac{1}{2}$, (for

the Sine of 30 is half the Chord of 60°, which is $\frac{1}{6}$ of the whole Circle, and is the Side of a Hexagon equal to Radius.) Therefore $AD^2 = AC^2 - CD^2 = 1 - \frac{1}{4} = \frac{3}{4}$; whence $AD = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$. And ſince $AD : CD :: AB : BH$, we have $\frac{1}{2}\sqrt{3} : \frac{1}{2} :: 1 : t = \frac{1}{\sqrt{3}}$; therefore $t^2 = \frac{1}{3}$, $t^3 = \frac{1}{3}\sqrt{\frac{1}{3}}$, $t^5 = \frac{1}{9}\sqrt{\frac{1}{3}}$, $t^7 = \frac{1}{27}\sqrt{\frac{1}{3}}$, &c. Laſtly, by ſubſtituting theſe Values of the ſeveral Powers of t in the above Equation, it gives the Arch $BC = \sqrt{\frac{1}{3}} \times 1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \frac{1}{729}$, &c.

200. Now as BC is $\frac{1}{6}$ of the Semicircle, we have $6\sqrt{\frac{1}{3}} = \sqrt{\frac{1}{12}} = 2\sqrt{3} = 3,4641016151$, &c. conſequently if this Number be multiplied by the ſeveral Terms of the Series $1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \frac{1}{729}$, &c. and the Sum of all the Negative Products taken from the Sum of all the Affirmative ones, the Remainder will be the Value of the Semicircle, whoſe Radius is Unity, viz. 3,1415926536; or this will be the whole Circumference, when the Diameter is equal to unity. The Reader may ſee the whole Operation in *Sherwin's Tables*, Page 55.

201. The whole Peryphery being known, any Part, Arch, or Angle, BAC becomes known, by the Rule of Proportion; and then the Sine CD of that Angle is alſo known. For by ſimilar Triangles ACD and ECG, we have $AD : AC :: EG : EC$, that is $\sqrt{1-x^2} : 1 :: \dot{x} : \dot{z} = \frac{\dot{x}}{\sqrt{1-xx}} = \dot{x} - \frac{1}{2}\dot{x}x^2 + \frac{3}{8}\dot{x}x^4 + \frac{5}{16}\dot{x}x^6$, &c. Therefore the Fluent $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 +$, &c.

202.

202. Having thus found the Value of z, we have $z^3 = x^3 + \frac{1}{2}x^5 + \frac{37}{120}x^7$; alſo $z^5 = x^5 + \frac{5}{6}x^7$; and laſtly, $z^7 = x^7$. Hence 'tis evident, that

$$\left\{\begin{array}{l} z - \frac{1}{6}z^3 = x \ast - \frac{1}{120}x^5 - \frac{1}{5040}x^7 \\ z - \frac{1}{6}z^3 + \frac{1}{120}z^5 = x \ast \ast + \frac{1}{5040}x^7 \\ z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{0640}z^7 +, \text{\&c.} = x = CD. \end{array}\right.$$

203. Suppoſe now the Arch $BC = z = 1 =$ Radius AB; then the Sine $CD = x = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} +$, &c. $=$ 0,84147098. Having thus obtained the Sine, the Co-ſine $AD = \sqrt{AC^2 - CD^2} = \sqrt{1 - 0{,}84147^2}$ is eaſily had; as alſo the Tangent BH, becauſe AD : CD : : AB : BH. And the Secant $AH = \sqrt{AB^2 + BH^2}$, or it is found by the Co-ſine as at (Art. 16).

204. The Number of Degrees in the Arch $z = 1$, is had by ſaying, as 3,14159, &c. : 2 : : 180° : 57°. 17′. 44″. And this is the famous Arch, or Angle, called the *Trigonometrical Modulus*, ſo much uſed in the *Harmonia Menſurarum* of the celebrated Mr. COTES.

205. By putting the Modular Arch 57° 17′ 44″ into Decimals, we have 57,29578 - - 1,758123

Whoſe Arithmetical Complement is - 8,241877

Suppoſe, now, the Sine of any other Angle were required, as that of 30″ 45′ = 30,75 in Decimals; then ſince $57{,}29578 : 1 : : 30{,}75 : \frac{30{,}75}{57{,}29578}$ it is evident,

206. That if to the Arith. Complement 8,241877
you add the Log. of the dec. Arch 30°,75 1,487845

The Sum is the Log. of the Arch in the *equal Parts of Radius* } 0,5367 9,729722

207. The Arch BC of 30°. 45′. being found equal to 05367=z, you find by the Series at (202.) the Value of x=0,5112931; ſee the Proceſs following.

Multiply the Logarithm of z	0,53669	9,729722
by 3	- -	3
		9,189166
Subduct the Logarithm of 6	- -	0,778151
Remains the Logarithm of $\frac{1}{6}z^3$=0,025764		8411015

Then $z-\frac{1}{6}z^3=0,53669-0,025764=0,510926$, which is *too ſmall*. Therefore take the third Term thus:

Multiply the Logarithm of z=0,53669		9,729722
by 5	- -	5
		8,648610
Subduct Logarithm of 120	- -	2,079181
Remains the Log. of $\frac{1}{120}z^5$=0,000371		6,569429

Then $z-{}^{1}z^3+\frac{1}{120}z^5=0,511297$, which is now a little too large; therefore, laſtly,

Multiply the Logarithm of z=0,53669		9,729722
by 7	- -	7
		8,108054
Subduct the Logarithm of 5040	-	3,702431
Remains the Log. of $\frac{1}{5040}z^7$=0,0000025		4,405623

So that $z-\frac{1}{6}z^3+\frac{1}{120}z^5-\frac{1}{5040}z^7=0,5112941=x$. Then the *Co-ſine* is AD=0,8594064; the *Tangent* BH=0,5949375 and the *Secant* AH=1,1635938; as they are to be ſeen in the Trigonometrical CANON.

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APPENDIX.

The PRINCIPLES of the *Stereographic* PROJECTION applied to the DELINEATION and SOLUTION of *Spherical* TRIANGLES, and the CONSTRUCTION of MAPS, DIALS, &c.

208. THE PLANE OF PROJECTION is that on which the Circles of the Sphere, great and ſmall, are projected, by Rays paſſing from every Point of them through the ſaid Plane, to the Eye; according to the Diſtance of the Eye, different Kinds of Projections will ariſe; as the *Orthographic*, the *Globular*, the *Stereographic*, &c.

209. The latter of theſe, viz. the *Stereographic* PROJECTION, is that which is principally concerned in *Spherical* TRIGONOMETRY; in this Projection we conſider *three Sorts of Circles*, viz. (1.) A *direct Circle*, or one placed *directly before*, or *perpendicular to*, the Axis of the Eye. (2.) A *Right Circle*, or ſuch whoſe Plane paſſes through the Eye. (3.) An *Oblique Circle* is one in any other Poſition, than the two above mentioned.

210. Let ADEF (Fig. 26.) be a *right Circle*, whoſe Plane paſſes through the Eye at E, placed in its Periphery; draw the Diameters AE, DF at right Angles; and let EB be a Ray (or right Line) proceeding from the Point B to the Eye; this Ray will cut the Diame-

ter DF in the Point G; and becauſe the Point B, and all other Points in the Semi-periphery DAF are projected in like Manner into the Diameter DF, they are ſaid to be *Stereographically* projected, as the entire *ſolid Hemiſphere* is thus projected upon the Plane of that great Circle which is placed *direct before the Eye*, or whoſe Diameter is DF.

211. And thus a Point I, in the lower Semicircle DEF, is projected into the Point H, in the ſame Diameter FD continued out, beyond the Primitive ADEF.

212. On the Point E as a Center, with the Radius EC, deſcribe the Arch MC, and draw CB; then is the Angle $CEB = \frac{1}{2}ACB$ (by *Euclid's Elements*;) and as GC is the Tangent of the Angle CEG, ſo it is evident the Arch AB is projected into a right Line CG, which is the *Tangent of half the Arch*, and hence the Diameter FD has been called a LINE OF HALF-TANGENTS, or *Semi-tangents*.

213. Suppoſe B, I, the two Extremes of a right Circle BCI, placed obliquely to DCF, or making an Angle with it, BCD; then it is evident the Diameter of that oblique Circle will be projected (by the Eye at E) into the right Line GH.

214. Biſect GH in N, on which, as a Center, with Radius NG deſcribe the Circle AGELHQ; of which, the Part AGE is the Projection of that Half of the oblique Circle which is above the Primitive; and the other Part ELHQA is the Projection of the Semicircle below.

215. *The Diſtance of the Centers* CN *of the Primitive and projected Circles, is equal to the Tangent of the Elevation of the Oblique Circle above the Plane of the Primitive,*

that is, of the Angle BCD. For join EN, and with Radius EC ſtrike the Arch CK, cutting EN in K. Then (becauſe of the right-angled Triangle GEH, (in a Semicircle GLH,) we have the Angle GEC=EHC; but GEC=½ACB, and EHC=½ENC; therefore ENC=ACB; conſequently CEN=BCD, or the Arch CK=BD; but CN is the *Tangent* of CK, and therefore of BD, the *Elevation of the Oblique Circle*, which was to be ſhewn.

216. It is likewiſe at the ſame time evident, that EN=GN, is the *Secant* of the ſame Angle CEN or BCD; therefore the *Secant of the Elevation ſet* from G, or the *Tangent from the Center* C, finds the Center N of the given Oblique Circle to be projected; and hence the *Line* DH is called the *Line of Meaſures*, for all Projections on the Primitive ADEF.

217. If BI be the Diameter of a *ſmall Circle*, parallel to AE the Diameter of a large one (Fig. 27.); the center of its Projection N will be diſtant from the Center of the Primitive C by the *Secant of its Diſtance* BF *from the neareſt Pole* F *of the Primitive*; and *its Radius will be equal to the Tangent of that Diſtance.* For by reaſon of the ſimilar Triangles GBM and HBM, right angled at M, we have the Angle GBM=BHM=NBH; but GBM=GEC=GBC; therefore GBC=NBH. If then from the right Angle GBH you deduct on one Side the Angle NBH, and to the Remainder add the equal Angle CBG, on the other, you will have the right Angle CBN. Therefore BN the Radius the Projected Circle, is the *Tangent* of the Angle BCF, and CN the *Secant* thereof. Q. E. D.

218. To find the Pole of an Oblique Circle AGE (Fig. 26.) Through G draw the Line EGB; and

from the Point B ſet off 90 Degrees to P; then from P lay a Ruler to E, and it will croſs the Diameter DF in O, the Pole required. Or thus, the Tangent of $\frac{1}{2}$DB ſet from C to O, gives the Pole O as before, becauſe AP=BD.

219. It is an univerſal Principle in *Stereographics*, that a ſtrait Line OH drawn from the Pole O of any oblique Circle to the Primitive, will cut off an Arch AB from the ſaid oblique Circle equal to the Arch AH in the Primitive, which meaſures the Angle ACH. For (1.) Suppoſe the Pole O were in the Center C, then the *oblique Circle* AGE would become the *direct Circle* AFE, or coincide with the Primitive, and the Arch AB would become AH. (2) Again, ſuppoſe the Pole O were placed in the Primitive at D, then would the oblique Circle AGE become the right Circle ACE; and the Line DH will cut off a Part A*b* equal to AH, or Angle ACH (by Art. 212.) as before. Therefore (3.) all the while the Pole O is advancing from the Periphery at D to the Center at C, the Circle AGE will be *oblique*, and the Line OH will conſtantly cut off a Part AB=AH in the Periphery. (See Fig. 28.)

220. *The Diſtance between the Poles of any two Circles in the Projection is equal to the Angle which is formed by their Planes at their interſection.* Thus CO the Diſtance between the Pole C of the Primitive, and the Pole O of the oblique Circle AGE, is equal to AQ=PF=FG, the Meaſure of the Angle of Inclination FAG. (by Art. 212). See Fig. 28.

221. By what has been demonſtrated (at Art. 215.) it is eaſy *to draw an Oblique Circle through any given Point* P *in the Plane of Projection to make a given Angle* GIK with the Primitive Circle. For if with the *Tangent* of

the given Angle, and one Foot of the Compaſſes in C, you ſweep the ſmall Arch (*ab*;) and then with the *Secant* of the ſaid Angle, and one foot in the given Point P, you ſweep another ſmall Arch (*cd*) to croſs the former in (*e*); then will (*e*) be the Center, on which to draw the Oblique Circle IGL to make the given Angle GIK required. Fig. 30.

222. *To conſtitute an Angle of a given Quantity between any two Circles.* (1.) Let the Angle be at the *Center* C of the Primitive. Then having adapted the Sector to the Radius AC (Fig. 29.) take from the *Chords* the Number of Degrees in the given Angle, and ſet them from A to H, and draw CH; and the Angle ACH is that required.

223. (2.) Let the Angle be at the *Periphery* of the Primitive, as at A, (Fig. 26.) Then drawing the two Diameters AE and DF at right Angles, take the Tangent of the given Angle from the Sector, and ſet it from C to N; and upon N, with the Radius AN, deſcribe the Circle AGE, and it will make the Angle DAG required, (by 215.)

224. (3.) When the Angle is neither at the Center nor Periphery, but at ſome Point P, contained between the Arch of a right Circle PE, and of an oblique one PB. (Fig. 29.) Here you proceed thus; ſet the Tangent of AP from C to N; and upon N, with Radius NP, deſcribe the Circle LPMS for a *new Primitive*; and then LM will be a *new Line of Meaſures*; and ſince the Angle BPE is given, the Angle DPB, or LPH, is given; therefore (by Art. 221.) you draw the oblique Circle HP to make with the Primitive PL,

the given Angle LPH; and then its Complement HPN is the Angle made, as required.

225. (4.) Let a given oblique Circle P B be interſected in P by another oblique Circle PG, under a given Angle DPG; then in the new Primitive make the Angles HPN, and IPN (by Article 221.) ſuch that their Difference may be the Angle given HPI, and the Thing is done.

226. By this famous Problem all the MERIDIANS of the GLOBE or SPHERE are drawn in MAPS, and *Diagrams*, for DIALLING, &c. in that which is uſually called the *Horizontal* PROJECTION. Alſo by compleating the new Primitive LPM, and all the Meridians PH, PI, &c. You have the *common* MAPS of the WORLD in two HEMISPHERES, upon the Plane of the *General* or *Firſt* MERIDIAN. I ſhall only obſerve further, that if the Radius LN be divided into 9 equal Parts, and through theſe and the Poles P and S oblique Circles (as PHS, PIS, &c.) are drawn, they will be the MERIDIANS through each 10 Degrees, and becauſe they are equidiſtant in the EQUATOR LM, as they are on the GLOBE itſelf; therefore ſuch a MAP is ſaid to be on the *Globular* PROJECTION, and is the beſt of all others. Alſo the *ſmall Circles* or PARALLELS are drawn equidiſtant in the Line PS, and in the Periphery of the Primitive, in the ſame Manner.

227. From the Principles premiſed it will not be difficult for the young Trigonometer to *project any Caſe of Spherical Triangles* he may have Occaſion for, as in the following Examples. In a *right-angled Spherical* TRIANGLE let there be given the Baſe or one Side 36°. 15′. and the adjacent Angle 42°. 34′. to find the other Parts,

by Projection. Deſcribe the Primitive ADEF (Fig. 31.) and its two Diameters AE,DF; then ſet off 36°. 15′. from A to H, from the Chords on the Sector fitted to the Radius CA. And (by Art. 221.) draw the Oblique Circle AGE to make an Angle GAF=42° 34′ with the Primitive: and, laſtly, draw the Radius CH, and the propoſed Triangle ABH, right-angled at H, is projected, as required.

228. Now to find the *Side* BH, a Part of a right Circle, it is always ſuppoſed (in theſe Projections) that the Radius AC is equal to a *Line of Half Tangents* you have on your Rule; then on that Line ſet off the Diſtance CB, and it will be found to contain 61°. 30′. whence its Complement BH is 28°. 30′. the Side required.

229. To find the *Hypothenuſe* AB; firſt find the Pole O of the Oblique Circle AGE (by Art. 218.) then lay a Ruler from O to B, it will cut the Primitive in *(a)*, then the Arch A*a* meaſured on the Line of Chords, will be found 44°. 52′, for the Side AB, as required (219.)

230. To find the Angle ABH. Set 90 Degrees from H to P; and the Point P will be the Pole of the right Circle HI; but the Pole of the oblique Circle AGE is O; therefore lay a Ruler from the Angle B to the Pole O, and it will cut the Primitive in (*b*); then the Arch P*b*, meaſured on the Chords, gives the Angle ABH= 56°. 57′.

231. It is eaſy to ſee that the Triangle BCG, at the Center, is the Complement of that at the Periphery ABH; and that they are equally ſolvable by the ſame *Data*, and Proceſs of Operation.

232. Let a 2d. Example *be the Delineation of an oblique Spherical Triangle, wherein two Angles and a Side included are given.* Draw the Primitive, and its two Diameters AE, DF, (Fig. 32) and ſuppoſe the Triangle AHB, has the Angle at A=36°. 08′, the obtuſe Angle H=104° (or the Angle KHL=76°.) and the included Side AH=30° 00′. Then draw the oblique Circle AGE to make the Angle GAF=36°. 08′; and ſet off from the Chords AH=30°. 00′; then draw the Diameter HI and the oblique Circle HKI to make the Angle KHL=76° 00′; and thus will the Triangle AHB be projected as required.

233. Then, *to find the Side* AB; firſt find the Pole O of the oblique Circle AGE; and over the Points O and B lay a Ruler, and it will cut the Primitive in (*a*), then the Arch A*a*, meaſured on the Chords, will be found to be 42°. 09′.=AB, required.

234. To *find the Side* HB; find the Pole P of the oblique Circle HKI; and laying a Ruler from P to B, it will cut the Primitive in (*b*); then H*b* meaſured upon the Chords will be found 24° 04=HB, required.

235. To find the Angle at B; a Ruler laid from B to the Poles O and P, will cut the Primitive in *c*, and *d*; then the Arch (*cd*) meaſured on the Chords will be found to contain 46° 18′=ABH, as required.

236. After all that has been premiſed, the whole THEORY of a HORIZONTAL DIAL (which is the moſt uſeful of all others) muſt be extremely obvious in every reſpect; for ſuppoſing in Fig. 29. the Primitive ADEF to be the HORIZON of any given Place; then AP is the LATITUDE of that *Place*, P is the Projection of the Pole; and KPB, QPG, &c. are Hour Circles of the

Sphere interſecting the Horizon in the Points K, Q, &c.

237. In order to find the Diſtances AK, AQ, &c. from the MERIDIAN AE, or Circle of XII, you have the right-angled Triangle APK in which the Side AP=Latitude, and the Angle APK=to the given Hour, are known; by which the Diſtance AK is known for any HOUR, *Half*, *Quarter*, or *Minute* if you pleaſe. Laſtly, by a Ruler laid over the Center C, and the Point K, you will draw the HOUR-LINES upon the *horizontal* PLAIN, as required.

238. As the young Trigonometer may not have always at hand a *compleat Stereographic Projection of the Sphere*, I have ſupplied him with one (in Plate IV.) upon the PLANE of the HORIZON, which will preſent moſt of the *Horological*, *Nautical*, and *Geographical Ideas*, which he ought to have always freſh in his mind.

239. For, *firſt*, it gives the *Projection* and *Poſition* of all the HOUR-CIRCLES, and their *Interſections* with the *Horizon* of LONDON, by which the *Hour-Lines* may be immediately drawn on ſuch a Plane; and the *Horizontal* DIAL made for that Metropolis.

240. *Secondly*, it preſents a View of the principal POINTS of the COMPASS, called the RHUMBS, with their NAMES; as alſo with a double HORIZON, with the Degrees of AZIMUTH from *North* and *South*; and thoſe of AMPLITUDE from *Eaſt* and *Weſt*.

241. *Thirdly*, the *Contours* or *Out-Lines* of all the CONTINENTS, viz. of EUROPE, ASIA, AFRICA, and AMERICA, are delineated; and a general Idea of *Longitude*, *Latitude*, *Hour of the Day*, *Seaſon* of the *Year*, *Length of Day and Night*, &c. may be eaſily obtained

from a View of it, for any given Part of the World with Regard to GREAT BRITAIN, by MEANS of the MERIDIANS, and PARALLELS of LATITUDE. Alſo the *Bearing*, or *Point of the Compaſs* on which any Part lies, from the CITY of LONDON in the Center.

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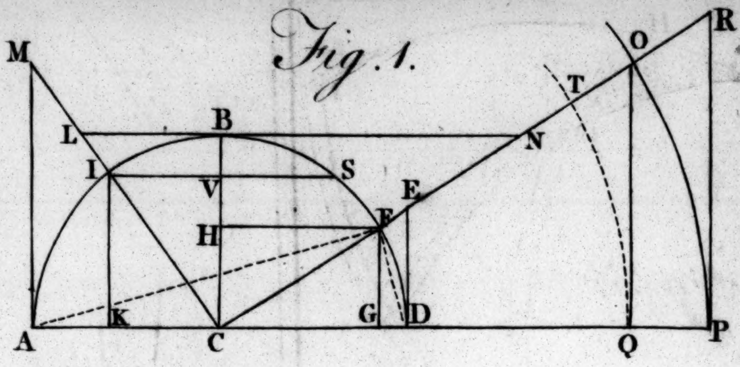
Fig. 1.



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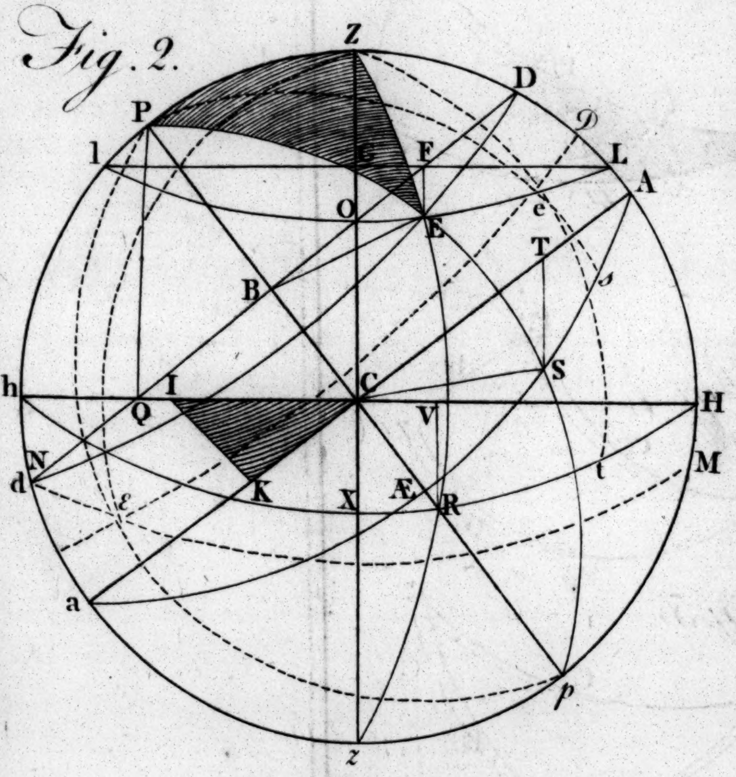

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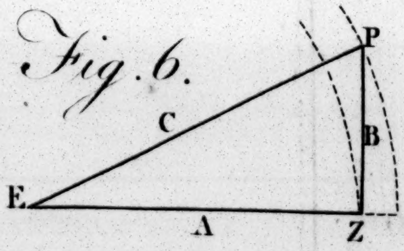

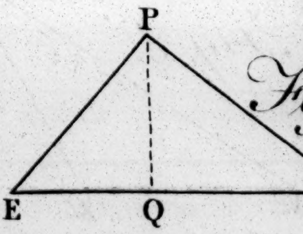

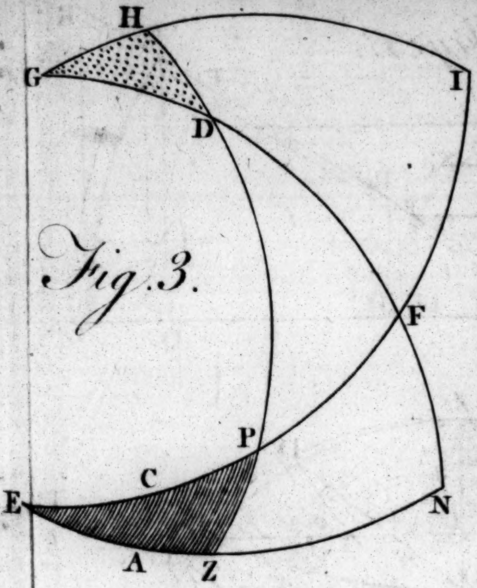

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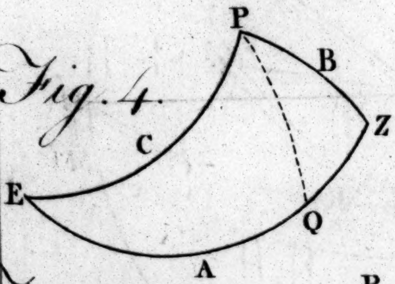

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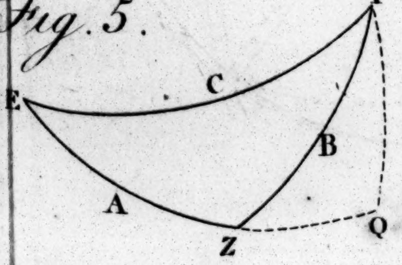

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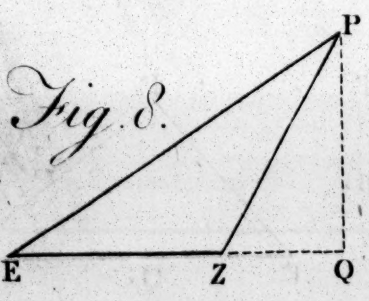

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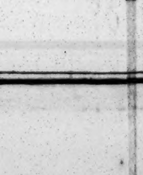

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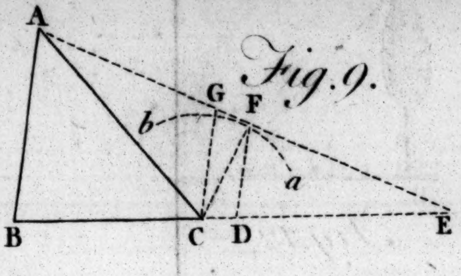

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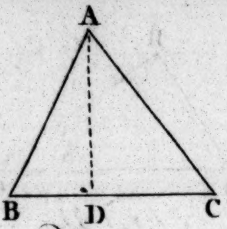

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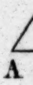

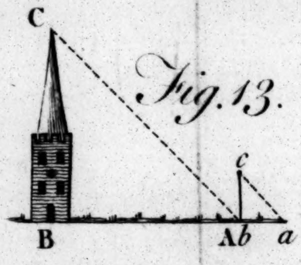

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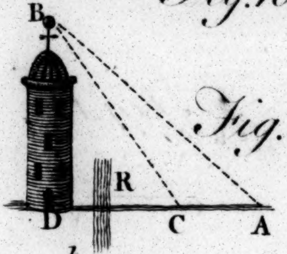

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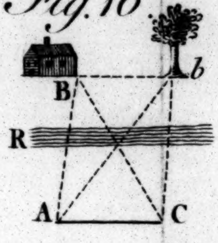
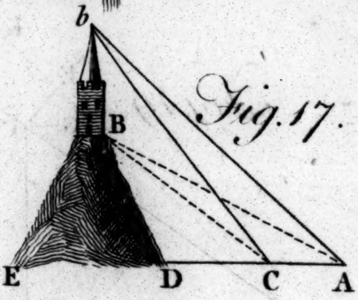
Fig. 16.





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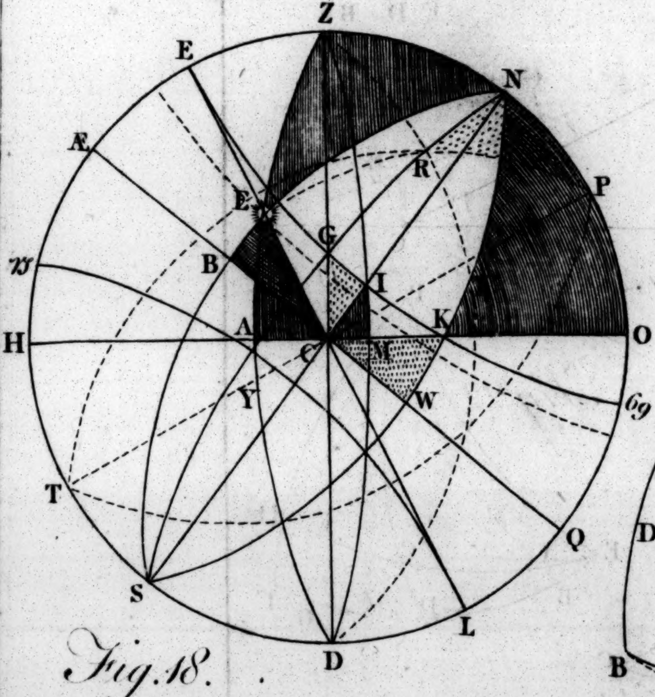

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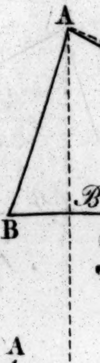

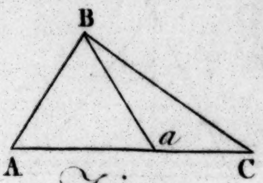

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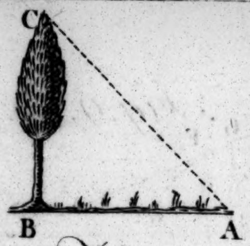

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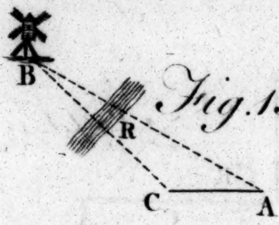

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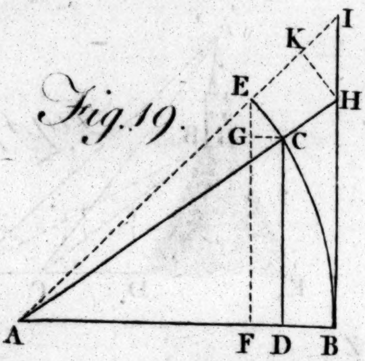

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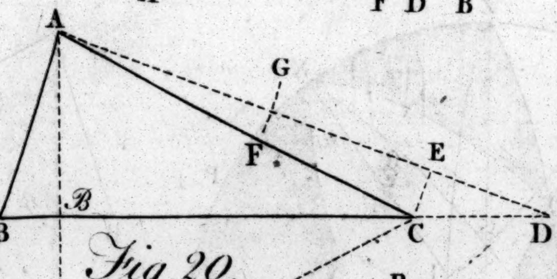

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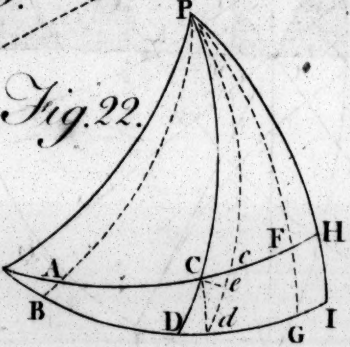

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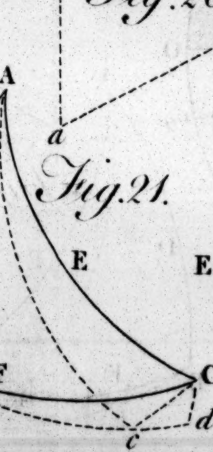

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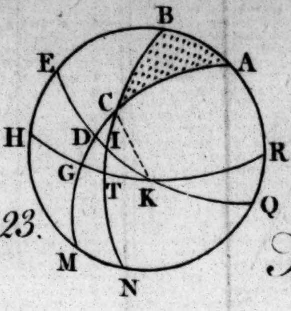
W.W. ſc.



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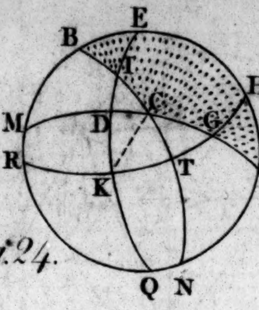

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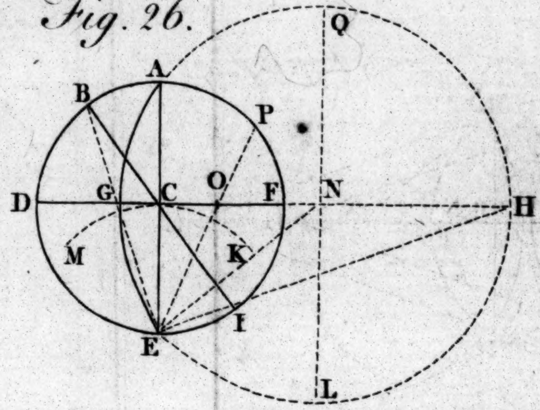

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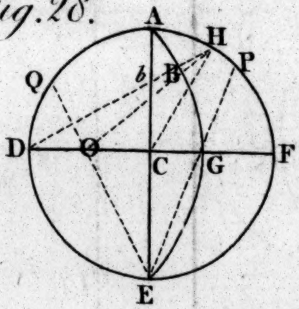

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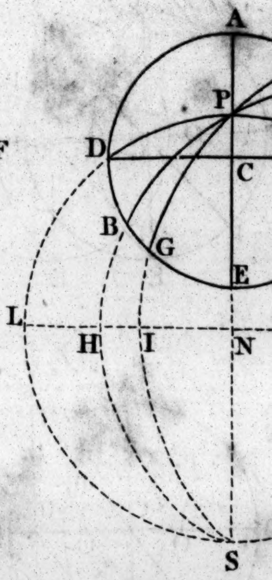

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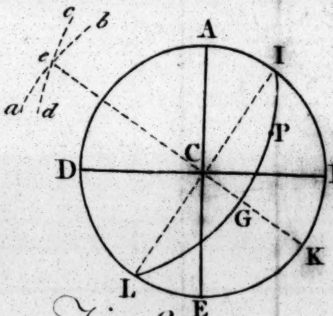

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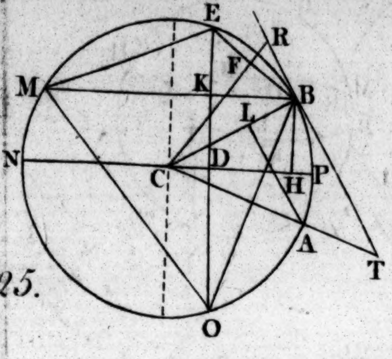

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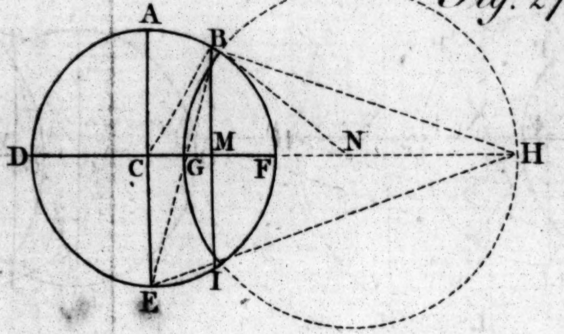

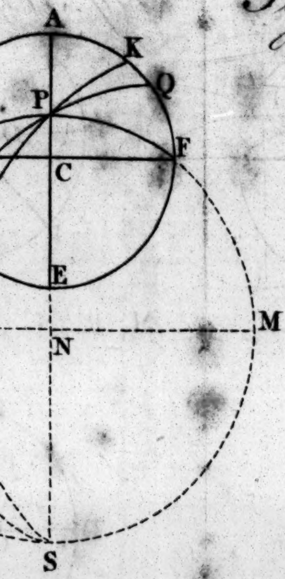

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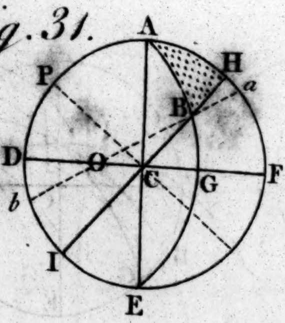

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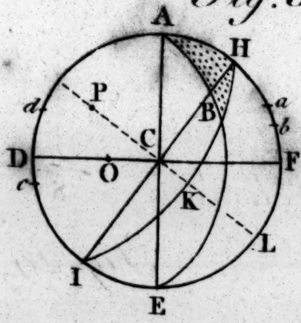

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